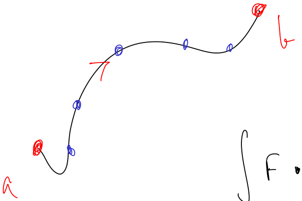
Last time: line integrals

Γ = some curve (oriented → spec'y dir of traversal).

$$F : \mathbb{R}^d \to \mathbb{R}^d$$

(only need $F : \Gamma \to \mathbb{R}^d$).



$$\int_\Gamma F \cdot d\ell = \text{line int of } F \text{ over } \Gamma$$

$$= \text{Work done by } F \text{ to move a body along } \Gamma$$

$$= \lim_{\|P\| \to 0} \sum F(\xi_i) \cdot (x_{i+1} - x_i)$$

Choose pts x_i along Γ (in dir of traversal)

$x_0 = a$, $x_n = b$

$\xi_i \in \Gamma$ between x_i, x_{i+1}

$$\|P\| = \max_i |x_{i+1} - x_i|$$

1 Compute $\int_\Gamma F \cdot d\ell$:

1. Choose a param $\gamma : [a, b] \longrightarrow \Gamma$

2. $$\int_\Gamma F \cdot d\ell = \int_a^b F \circ \gamma(t) \cdot \gamma'(t)\, dt$$

$$\left(\gamma(t) = \begin{pmatrix} \gamma_1(t) \\ \gamma_2(t) \\ \vdots \\ \gamma_d(t) \end{pmatrix} \text{ \& } \gamma'(t) = D\gamma(t) = \begin{pmatrix} \gamma_1'(t) \\ \gamma_2'(t) \\ \vdots \\ \gamma_d'(t) \end{pmatrix} \right)$$

3. Recitation: Param inv: If γ & δ are 2 param's of Γ (that traverse Γ in the same dir) then

$$\int_a^b F \circ \gamma(t) \cdot \gamma'(t)\, dt = \int_a^b F \circ \delta(t) \cdot \delta'(t)\, dt$$

Notation ① If Γ is a closed curve, then usually write

$$\int_\Gamma F \cdot dl = \oint_\Gamma F \cdot dl$$

② Often write $dl = \begin{pmatrix} dx_1 \\ dx_2 \\ dx_3 \end{pmatrix}$ & $F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$

write $\displaystyle\int_\Gamma F \cdot dl = \int_\Gamma F_1\, dx_1 + F_2\, dx_2 + F_3\, dx_3$

(Reason: Let $x(t)$ be a param of Γ. Then $\displaystyle\int_\Gamma F \cdot dl = \int_0^1 F(x(t)) \cdot x'(t)\, dt$

$$= \int_0^1 \sum \underbrace{F_i(x(t))}_{F_i(x)} \, \underbrace{x_i'(t)\, dt}_{dx_i} = \int_\Gamma \sum_i F_i(x)\, dx_i \quad)$$

(1) Arc-length integrals: $F : \mathbb{R}^d \to \mathbb{R}^d$, Γ a curve, defined $\int_\Gamma F \cdot dl \Rightarrow$ Work done.

$\hookrightarrow$ Want $\Gamma \subset \mathbb{R}^d$ some curve

$\rho : \mathbb{R}^d \longrightarrow \mathbb{R}$ the density of this wire.

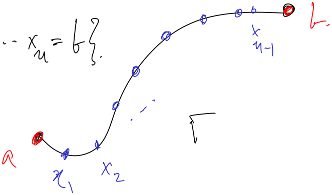
Q: Total mass?

① Notation: $\int_\Gamma \rho\, |d\ell|$ = arc len integral of ρ over Γ

$= \int_\Gamma \rho\, ds$ ($ds \to$ ref "arc len int")

② Intuition $\int_\Gamma \rho\, |d\ell|$ represents the total mass of the wire.

③ What is it: Choose pts $P = \{a = x_0, x_1 \cdots x_n = b\}$.

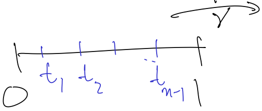


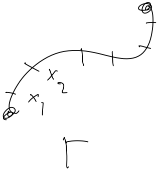
Approximate mass: If the wire was a straight line between x_i & x_{i+1} with const density then mass $= \sum_i \rho(x_i)\,|x_{i+1}-x_i|$

Expect actual mass $= \boxed{\int_\Gamma \rho\,|d\ell| = \lim_{\|P\|\to 0} \sum \rho(x_i)\,|x_{i+1}-x_i|.}$

(4) How do we compute??

(a) param Γ : Say $\gamma : [0,1] \longrightarrow \mathbb{R}^d$ is a param.





(b) $x_i = \gamma(t_i)$

$\sum \rho(x_i)\,|x_{i+1}-x_i| =$

$$\sum \rho(\gamma(t_i)) \, |\gamma(t_{i+1}) - \gamma(t_i)|$$

$$\overset{MVT}{=} \sum \rho(\gamma(t_i)) \, |\gamma'(\xi_i)(t_{i+1} - t_i)|$$

$$\Rightarrow \lim_{\|P\| \to 0} \sum \rho(x_i) \, |x_{i+1} - x_i| = \lim_{\|P\| \to 0} \sum \rho(\gamma(t_i)) \, |\gamma'(\xi_i)| (t_{i+1} - t_i)$$

$$= \int_0^1 \rho(\gamma(t)) \, |\gamma'(t)| \, dt$$

$$\therefore \int_\Gamma \rho \, |dl| = \int_0^1 \rho \circ \gamma(t) \, |\gamma'(t)| \, dt$$

Formula to compute.

Note: arc len integrals are direction independent.

Note: If $\Gamma \subseteq \mathbb{R}^d$ is a curve, then $\int_\Gamma 1 \, |d\ell| = \text{arc length}(\Gamma)$.

$\boxed{\text{Goal 2:}}$ Fundamental thm of line integrals.

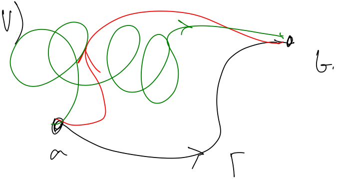
① Say $V : \mathbb{R}^d \to \mathbb{R}$ is C^1 (V is the "potential")

Let $F = -\nabla V$

② Let $\Gamma \subseteq \mathbb{R}^d$ be any curve starting at a & ending at b.

③ Then: $\int_\Gamma F \cdot d\ell = V(a) - V(b) = -(V(b) - V(a))$.

Cor: If F is a potential force (i.e. $F = -\nabla V$) then $\int_\Gamma F \cdot dl$ only depends on the start & end pts of Γ & not the path taken !!



Pf of theorem: ① Let γ be a param of Γ

② $$\int_\Gamma F \cdot dl = \int_0^1 F \circ \gamma(t) \cdot \gamma'(t)\, dt = -\int_0^1 \nabla V(\gamma(t)) \cdot \gamma'(t)\, dt$$

Compute $\frac{d}{dt}\left(V(\gamma(t)) \right) \overset{\text{chain rule}}{=} \sum_i \partial_i V(\gamma(t))\, \gamma_i'(t)$

$$= \underbrace{\nabla V(\gamma(t)) \cdot \gamma'(t)}$$

$$\Rightarrow \int_\Gamma F \cdot dl = -\int_0^1 \frac{d}{dt}\left(V(\gamma(t)) \right) dt \overset{FTC}{=} -\left(V(\gamma(1)) - V(\gamma(0)) \right)$$

$$= V(a) - V(b)$$

QED.