21-268: Multidimensional Calculus **Spring 2020**

Recitation April 7

Example of Coordinate Change

Recall the change of variables:

Suppose $U, V \subset \mathbb{R}^3$, and $\varphi : U \to V$ is C^1 and bijective. Then,

$$\int_V f(x)\,\mathrm{d}x = \int_U (f \circ \varphi)(y) \cdot |\det(D\varphi_y)|\,\mathrm{d}y$$

Problem

Compute the volume of the ellipsoid $V = \{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1\}$.

Solution Define $\psi : U \to V$ as $\psi : (\rho, \theta, \varphi) \mapsto (a\rho\cos\theta\sin\varphi, b\rho\sin\theta\sin\varphi, c\rho\cos\varphi)$, where $\rho \in [0, 1]$, $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$, and so $U = \{(\rho, \theta, \varphi) : \rho \in [0, 1], \theta \in [0, 2\pi], \varphi \in [0, \pi]\}$. Note that ψ is C^1 and bijective. We compute

$$\begin{aligned}
\det(D\psi) &= \begin{vmatrix} a\cos\theta\sin\varphi & -a\rho\sin\theta\sin\varphi & a\rho\cos\theta\cos\varphi \\ b\sin\theta\sin\varphi & b\rho\cos\theta\sin\varphi & b\rho\sin\theta\cos\varphi \\ c\cos\varphi & 0 & -c\rho\sin\varphi \end{vmatrix} \\
&= c\cos\varphi(-ab\rho^2\sin^2\theta\sin\varphi\cos\varphi - ab\rho^2\cos^2\theta\sin\varphi\cos\varphi) \\
&\quad - c\rho\sin\varphi(ab\rho\cos^2\theta\sin^2\varphi + ab\rho\sin^2\theta\sin^2\varphi) \\
&= abc\rho^2(-\sin^2\theta\sin\varphi\cos^2\varphi - \cos^2\theta\sin\varphi\cos^2\varphi - \cos^2\theta\sin^3\varphi - \sin^2\theta\sin^3\varphi) \\
&= abc\rho^2(-\sin\varphi\cos^2\varphi - \sin^3\varphi) \\
&= -abc\rho^2\sin\varphi
\end{aligned}$$

Thus, we can compute the volume

$$\begin{aligned}
\int_V 1\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z &= \int_U 1 \cdot |\det(D\psi)|\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= \int_0^\pi \int_0^{2\pi} \int_0^1 abc\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= abc\int_0^\pi \int_0^{2\pi} \frac{1}{3}\sin\varphi\,\mathrm{d}\theta\,\mathrm{d}\varphi \\
&= abc\int_0^\pi \frac{2\pi}{3}\sin\varphi\,\mathrm{d}\varphi \\
&= \frac{2\pi}{3}abc \cdot (-\cos\varphi)_{\varphi=0}^{\pi} = \frac{4\pi}{3}abc
\end{aligned}$$

Example of a Line Integral

Recall how we compute a line integral using its parametrization:

Let $\gamma : [0,1] \to \mathbb{R}^d$ be a parametrization of the curve Γ,

$$\int_\Gamma F \cdot dl = \int_0^1 (F \circ \gamma)(t) \cdot \gamma'(t) dt$$

Now we look at the following example in $\mathbb{R}^3$:

Problem

Compute

$$\int_\Gamma yzdx + xzdy + xydz = \int_\Gamma \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} \cdot dl$$

along the curve

$$\Gamma = \{(t, t^2, t^3) | t \in [0,1]\}$$

Solution Here we are given an parametrization $\gamma : [0,1] \to \mathbb{R}^3$ via

$$\gamma(t) = (t, t^2, t^3)$$

Plug in the formula with γ and $\gamma'(t) = (1, 2t, 3t^2)$ we get

$$\begin{aligned} \int_\Gamma \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix} \cdot dl &= \int_0^1 \begin{pmatrix} t^5 \\ t^4 \\ t^3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix} dt \\ &= \int_0^1 6t^5 dt \\ &= t^6 |_{t=0}^1 = 1 \end{aligned}$$

Parametrization Invariance of Line Integrals

Proposition 1 *Let $\gamma : [0,1] \to \mathbb{R}^d$ and $\delta : [0,1] \to \mathbb{R}^d$ be two parametrizations of the curve Γ. Suppose that there exists a C^1 bijective function $\varphi : [0,1] \to [0,1]$ such that $\varphi(0) = 0$, $\varphi(1) = 1$, and $\delta = \gamma \circ \varphi$. Then,*

$$\int_0^1 (F \circ \delta)(t) \cdot \delta'(t) \, \mathrm{d}t = \int_0^1 (F \circ \gamma)(s) \cdot \gamma'(s) \, \mathrm{d}s$$

Proof: By computation,

$$\begin{aligned}\int_0^1 (F\circ\delta)(t)\cdot\delta'(t)\,\mathrm{d}t &= \int_0^1 (F\circ(\gamma\circ\varphi))(t)\cdot(\gamma\circ\varphi)'(t)\,\mathrm{d}t\\ &= \int_0^1 ((F\circ\gamma)\circ\varphi)(t)\cdot\gamma'(\varphi(t))\cdot\varphi'(t)\,\mathrm{d}t \quad \text{(by chain rule)}\\ &= \int_0^1 (F\circ\gamma)(\varphi(t))\cdot\gamma'(\varphi(t))\cdot\varphi'(t)\,\mathrm{d}t\\ &= \int_0^1 (F\circ\gamma)(s)\cdot\gamma'(s)\,\mathrm{d}s \quad \text{(by change of variable } s=\varphi(t))\end{aligned}$$

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Line Integral w.r.t Arc Length

In usual integral we integrate a scalar-valued function in a domain. In line integral since we are integrating things in a curve instead of an open set in $\mathbb{R}^d$, we integrate the dot product with a vector field.

However recall that a parameterized curve can be view as a 1-dimensional object just like $[0,1]\subset\mathbb{R}$, one might want to develop some form of line integral similar to the integration in $\mathbb{R}$. We will see how through the following example.

Problem

Calculate the perimeter of a unit half circle Γ using line integral

Solution

Let $\gamma:[0,1]\to\mathbb{R}^2$ be the parametrization of Γ via $\gamma(t)=(\cos(\pi t),\sin(\pi t))$ This problem is the same as asking "integral 1 over the half circle". In an usual integral with vector field F over a curve with parameterization γ, we are summing up small parts of $F\cdot\gamma'$. Here we want to sum up small part that represent the length, i.e. $|\gamma'|$. We want to have an F such that $(F \circ \gamma) \cdot \gamma' = \gamma'$, so we find some F such that $F \circ \gamma = \dfrac{\gamma'}{|\gamma'|}$. Then the integral becomes

$$\begin{aligned}
\int_\Gamma F \cdot dl &= \int_0^1 (F \circ \gamma)(t) \cdot \gamma'(t) dt \\
&= \int_0^1 \frac{\gamma'(t)}{|\gamma'(t)|} \cdot \gamma'(t) dt \\
&= \int_0^1 |\gamma'(t)| dt \\
&= \int_0^1 \sqrt{(-\pi \sin(\pi t))^2 + (\pi \cos(\pi t))^2} dt \\
&= \int_0^1 \pi dt \\
&= \pi
\end{aligned}$$

as expected.

In general for any scalar valued function $f : \Gamma \to \mathbb{R}$, curve Γ with parametrization $\gamma : [0, 1] \to \Gamma$, we write the integration of f with respect to the arc length as

$$\int_\Gamma f ds = \int_0^1 (f \circ \gamma)(t) |\gamma'(t)| dt$$

In particular when $f = 1$, we get the arc length of Γ.