$$\underset{U}{(u,v)} \xrightarrow{\varphi} \underset{V}{(x,y)} \xrightarrow{f} \mathbb{R}$$

$\varphi : U \longrightarrow V$ bij

$$\int_V f(x)\,dx = \int_U (f \circ \varphi(y))\ |\det D\varphi(y)|\ dy$$

$U \to \varphi^{-1}(V) \leftarrow$ new domain.

Q5 : $$\int_{\mathbb{R}^2} \underbrace{\frac{1}{y^2+4\sqrt{x^2+y^2}+4} \; \frac{1}{\sqrt{x^2+y^2}}}_{f(x,y)} \, dx\, dy$$

You write f in terms of u & v.

$$\varphi(u,v) = \begin{pmatrix} u^2-v^2 \\ 2uv \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$D\varphi = \begin{pmatrix} 2u & -2v \\ 2v & 2u \end{pmatrix} \Rightarrow |\det D\varphi| = 4(u^2+v^2)$$

$$\Rightarrow \int_{\mathbb{R}^2} f(x,y)\, dx\, dy = \int_{??} f\circ\varphi(u,v)\; 4(u^2+v^2)\; \underbrace{du\, dv}_{dA}$$

$??$: U, $\mathbb{R}^2$

Domain of the u-v integral $=$ U

$$\left(\begin{matrix} x \\ y \end{matrix}\right) = \varphi \left(\begin{matrix} u \\ v \end{matrix}\right)$$

$$\int_{\mathbb{R}^2} f \, dx\,dy = \int_U f \circ \varphi \; |\det D\varphi| \; du\,dv$$

(chosen so that $\varphi : U \to \mathbb{R}^2$ is C^1 & bijective)

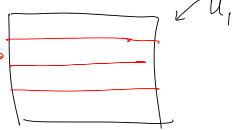
If $\varphi : U \longrightarrow \mathbb{R}^2$ is ~~C~~ bij then $\underbrace{U = \varphi^{-1}(\mathbb{R}^2)}_{\text{can't say this!}}$??

(Polar coordinates: $x = r \sin\theta$, $y = r\cos\theta$ | $\varphi(r,\theta) = \left(\begin{matrix} x \\ y \end{matrix}\right)$.

$$\underbrace{\varphi^{-1}(\mathbb{R}^2) = \{(r,\theta) \mid r\in\mathbb{R},\ \theta\in\mathbb{R}!!\}}_{U_1}$$

$\varphi : U_1 \to \mathbb{R}^2$ is not bij!!

For change of coord:



U

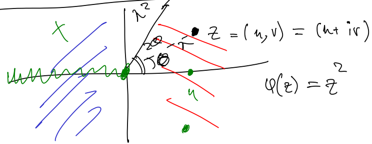
$\varphi : U \to \mathbb{R}^2$ is bij

$$\int_{\mathbb{R}^2} f\,dA = \int_U f\circ\varphi\,|\det D\varphi|\,dA$$

(for each strip).

Ex $\varphi(u,v) = \begin{pmatrix} u^2 - v^2 \\ 2uv \end{pmatrix}$. Need to find U + $\varphi : U \to \mathbb{R}^2$ is C^1, bij.

$\varphi(-u,-v) = \varphi(u,v)$. Could try $\boxed{U = \{(u,v) \mid u > 0\}}$



$(u-v)^2 = u^2 + v^2 - 2uv$

$$(u+iv)^2 = \underbrace{u^2 - v^2}_{\text{1st coordinate of } \varphi} + i\underbrace{(2uv)}_{\text{2nd coordinate of } \varphi}$$

Think of $\mathbb{R}^2 = \mathbb{C}$ (complex plane)

$\varphi(z) = z^2$

$z = x + iy$

intuition for me to help vis.

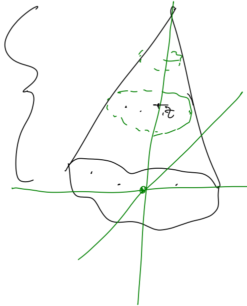
$z = u + iv$

$\varphi(u,v) = \boxed{z^2 \to \text{Real} + i\,\text{img}} \to (u^2 - v^2,\ 2uv)$

$$\int_{-\infty}^{\infty}\int_{u=0}^{\infty} f\circ\varphi \; |\det D\varphi| \, dA = \int_{\mathbb{R}^2} f \; dA$$

Q4:

h



$U \subseteq \mathbb{R}^2$

$$C = \left\{ (x,y,z) \;\middle|\; 0 < z < h, \; \left(\frac{hx}{h-z}, \frac{hy}{h-z}\right) \in U \right\}$$

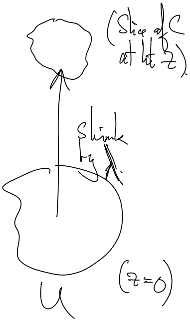
$$\mathrm{Vol}(C) = \int_C 1 \; dx\,dy\,dz$$

$$= \int_{z=0}^{h} \left(\int_{(x,y,z)\in C} 1 \; dx\,dy \right) dz$$

$$= \int_{z=0}^{h} \text{area}\left(\text{slice of } C \text{ at height } z\right) \, dz$$

$$= \int_{z=0}^{h} \text{area}(U) \cdot \underbrace{\left(\text{scaling factor}\right)}_{\lambda^2} \, dz$$

$$= \int_{z=0}^{h} \text{area}(U) \, \Big(\underbrace{1 - \frac{z}{h}}_{\lambda}\Big)^2 \, dz$$



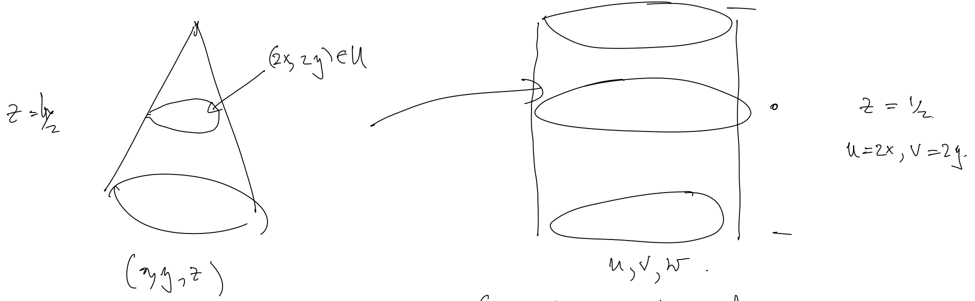
Formula for λ: $z = 0 \Rightarrow \lambda = 1$

$z = h \Rightarrow \lambda = 0$

λ is linear in between!

Make rigorous!

Coordinate change from limit: $u = \frac{hx}{h-z}$, $v = \frac{hy}{h-z}$, $w = z$



(C becomes the cylinder $U \times (0,h)$ in coordinates u, v, w.)

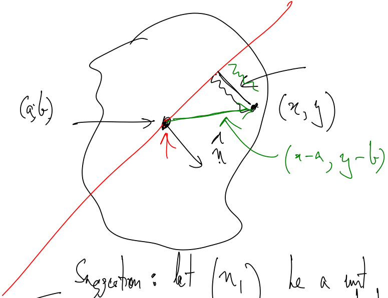
Compute $\mathrm{Vol}(C) = \int_C 1 \, dx\,dy\,dz = \int_{w=0}^{h} \left(\int_{(u,v)\in U} |\mathrm{Det}\, D\varphi| \, du\,dv \right) dw$ (u,v,w)

$$= \int_{w=0}^{h} \left(\int_{(u,v)\in U} \left(1 - \frac{w}{h}\right)^2 du\,dv \right) dw$$

$$= \int_{w=0}^{h} \mathrm{area}(U) \left(1 - \frac{w}{h}\right)^2 dw$$ (what we guessed before!).

$$a = \frac{1}{M}\int_D x \; \rho(x,y)\; dA$$

$$b = \frac{1}{M}\int_D y \; \rho(x,y)\; dA.$$



Compute T_ℓ = torque about ℓ.

$$\int_D \rho(x,y)\; \underbrace{d(x,y)}_{\perp \text{ dist}} \; dA$$

Suggestion: let $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$ be a unit normal to ℓ

Compute d: $(x-a, y-b) \cdot \hat{n}$