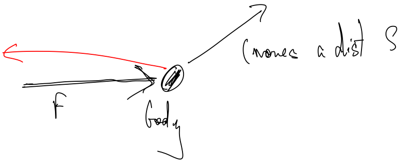
Line integrals, Surface integrals & Stokes theorems.

① Line integrals: Physics motivation → work done by a force.

Apply a force of mag F
the body moves a dist S
Work done by the force $= F \cdot S$

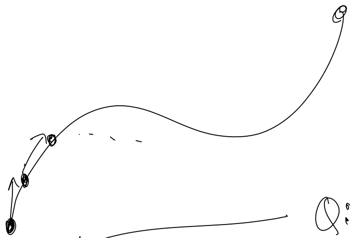


Need F times dist moved in the dir of S.

<u>Def</u>: Apply a force F (vector) to a body B.
Displacement as a result of the force is Δr (vector).

Work done by the force is defined to be $\underbrace{F}_{\text{vect}} \underbrace{\cdot}_{\text{dot product.}} \underbrace{(\Delta r)}_{\text{vect}}$

Our case :

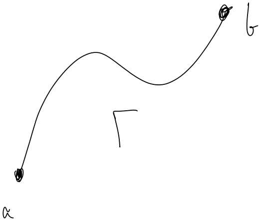


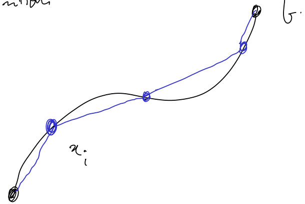
Push a body continuously with a varying force.
Body traces out a curve Γ

Q: What is the work done??

① Approximate Γ by line segments.

Approximation





(1) Let $x_0, x_1, \dots x_n$ be n points on the curve. with $x_0 = a$ & $x_n = b$.

(2) Work done is approximately $\sum_{i=0}^{n-1} F(x_i) \cdot (x_{i+1} - x_i)$

Def: Let $\Gamma \subseteq \mathbb{R}^d$ be a curve. Let $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a fn (force)

Specify the starting pt of $\Gamma = a$ } $\leftarrow$ (Specifies a direction of traversal)
& ending pt of $\Gamma = b$

Let $P = \{x_0, \dots x_n\}$ $\ni$ $x_i \in \Gamma$ & $x_0 = a$, $x_n = b$.

Define the line integral of F along Γ by

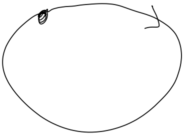
$$\underbrace{\int_\Gamma F \cdot dl}_{\text{line int.}} \stackrel{\text{def}}{=} \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} F(x_i) \cdot (x_{i+1} - x_i)$$

$(\|P\| = \text{mesh size of } P = \max_i |x_{i+1} - x_i|)$

(Intuition: $\int_\Gamma F \cdot dl$ = work done by F to move a body along Γ

Remark: ① Specifying end points → specifying a dir of traversal.

Closed curves:



Start & end pt are the same.
Need to indicate a direction of traversal (orientation)

2D: Orient by specifying clockwise (CW) or counter clockwise (CCW).

← 3D CW/CCW don't make sense (need something else).

Remark ② Piecewise C^1 curves.

So far: all curves for us were C^1 curves (no corners)

Piecewise C^1 $\Rightarrow$ allows for finitely many corners (no cusps).

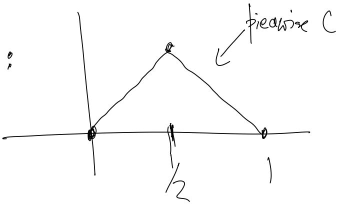
Def: A fn $f:[0,1] \to \mathbb{R}^d$ is called <u>piecewise</u> C^1 if:

① $\exists$ finitely many $x_0, x_1, \dots x_n$ s.t. $x_0 = 0 < x_1 < \cdots < x_n = 1$.

② The fn $f:[x_i, x_{i+1}]$ is C^1.

③ & $f:[0,1] \to \mathbb{R}^d$ is cts.

Eg:



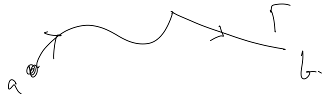
Q: How do we compute line integrals?

$\lim_{\|P\| \to 0} \sum F(x_i) \cdot (x_{i+1} - x_i)$ too troublesome to compute with specifically.

$\longrightarrow$ Reduce line integrals to a std Riemann int.

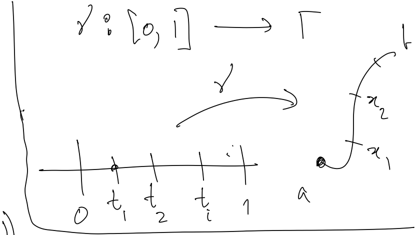
Let $\Gamma \subseteq \mathbb{R}^3$ be a piecewise C^1 curve (starting from a ending at b)

Let $\gamma : [0, 1] \longrightarrow \Gamma$ be a param (piecewise C^1, $\gamma' \neq 0$, bij)



Let $t_0 = 0 < t_1 < \cdots < t_n = 1$

Let $x_i = \gamma(t_i)$



Note
$$\int_\Gamma F \cdot d\ell = \lim_{\|P\| \to 0} \sum F(x_i) \cdot (x_{i+1} - x_i)$$
$$= \lim_{\|P\| \to 0} \sum F \circ \gamma(t_i) \cdot \left(\gamma(t_{i+1}) - \gamma(t_i)\right)$$

MVT: $\exists\, \xi_i \in (t_i, t_{i+1}) \ni$
$\gamma(t_{i+1}) - \gamma(t_i) = \gamma'(\xi_i)\,(t_{i+1} - t_i)$
$(\gamma' = D\gamma)$.

$$= \lim_{\|P\| \to 0} \sum F \circ \gamma(t_i) \cdot \gamma'(\xi_i)\,(t_{i+1} - t_i)$$
$$= \int_{t=0}^{1} F \circ \gamma(t) \cdot \gamma'(t)\, dt \longleftarrow \text{std Riemann int.}$$

Compute line int $\int_\Gamma F \cdot dl$.

① Param Γ (say γ is the param)

② $$\int_\Gamma F \cdot dl = \int_0^1 F \circ \gamma(t) \cdot \gamma'(t)\, dt$$

Domain of γ

(③ Make sure γ parametrizes Γ in the specified dir of traversal otherwise the answer will have ~~a~~ ~~a~~ the wrong sign!).

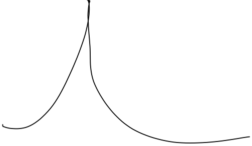
Q: last Wed: $I = \int_{-\infty}^{\infty} e^{-x^2}\,dx$

$$I^2 = \underbrace{\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)}_{\text{number}} \left(\int_{-\infty}^{\infty} e^{-\cancel{x}\,y^2}\,d\cancel{x}\,y\right)$$

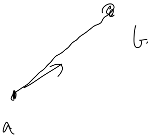
$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right) \underbrace{e^{-y^2}}_{\text{doesnt dep on } x}\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy$$

Cusp : Vertical tgts



At cusps $\quad |\gamma'| = +\infty$



Param tricks : ① Circle $\longrightarrow$ $x = r\cos t$
$y = r\sin t$

② line joining a & b : $\gamma(t) = a + t(b-a) = (1-t)a + tb$

③ write one coordinate in terms of the other & chose that to be t.