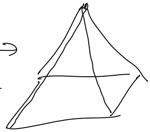
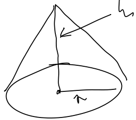
Q on HW (due Wed): Formula for volume of a cone? → $\frac{1}{3}\pi r^2 h$.

Pyramid.

Volume $= \frac{1}{3}$ area(base) height





← cone with a blob shaped base.

← Blob.

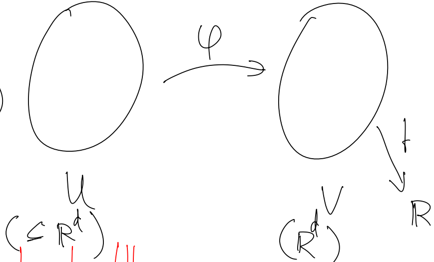
Volume is still $\frac{1}{3}$ (area base) height?

Yes! → Use change of var (2D) & prove (on HW).

Recall: Coordinate change.

Assumption: $\varphi : U \to V$ is C^1, bij (& $D\varphi_x$ is inv $\forall x \in U$)

(OK if this fails at finitely many points
OR finitely many nice curves
OR, 3D, " " " surfaces)



Claim

$$\int_V f(x)\,dx = \int_U f\circ\varphi(y)\,|\det D\varphi_y|\,dy$$

(area int in 2D
Vol int in 3D)

(Peano space filling curve → $\exists$ a fn : $[0,1] \to [0,1]^2$ $(\subseteq \mathbb{R}^2)$ which is cts and bij)

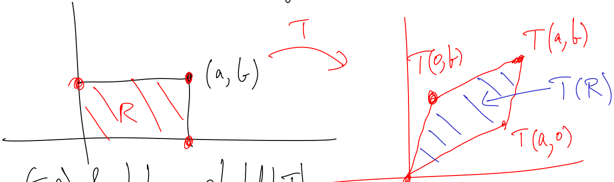
(i.e. Put $x = \varphi(y)$ in the int.
Change V to $\varphi^{-1}(V) = U$ & dx to $|\operatorname{Det} D\varphi|\,dy$)

Note ① Say $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear.

R = some rect in $\mathbb{R}^2$.

Q: area(TR) = area$\big(\underbrace{T(R)}_{\text{image of } R \text{ under } T}\big)$ = $|\det(T)|\,$area(R)

ⓐ stupid way to check.



Suffices & compute area(TR) & check $= ab\,|\det T|$

ⓑ better way to check:

① Write $T = T_1 T_2 \cdots T_n$ where T_i are elementary matrices

Elementary matrices (a) stretch along any coordinate axis

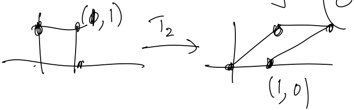
Eg $\begin{pmatrix}\lambda & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1\end{pmatrix}$

(b) Shears: Eg $\begin{pmatrix}1 & \alpha & 0 & 0 & 0 \\ & 1 & 0 & & \\ & & \ddots & \ddots & 0 \\ & & & & 1\end{pmatrix}$

(2) Area change factor if you stretch one axis by λ is: $= \lambda = \det(T_1)$

(3) " " " " " shear Eg $\begin{pmatrix}1 & 1 \\ 0 & 1\end{pmatrix} = T_2$ ∴ Area change factor $= 1$

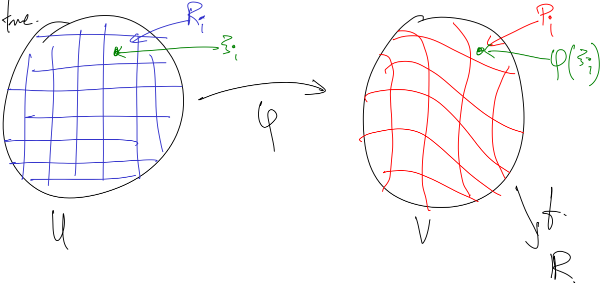
$\det(T_2) = 1$



$\Rightarrow \text{area}(TR) = |\det T| \cdot \text{area}(R).$

Why is the coordinate change formula true.

NTS $\displaystyle\int_V f(x)\,dx = \int_U f\circ\varphi(y)\,|\det D\varphi_y|\,dy$



Intuition: (1) Divide U into rectangles R_i

(2) Let $P_i = \varphi(R_i) \subseteq V$

(3) If R_i is small, what is area(P_i) approximately?

Let $\xi_i \in R_i$. Near ξ_i, $\varphi(x) \approx \varphi(\xi_i) + D\varphi_{\xi_i}(x - \xi_i)$

$\Rightarrow \varphi(R_i) \approx \varphi(\xi_i) + D\varphi_{\xi_i}(R_i - \xi_i) \Rightarrow \text{area}(\underbrace{\varphi(R_i)}_{P_i}) \approx |\det D\varphi_{\xi_i}|\,\text{area}(R_i)$

$\Rightarrow \text{area}(P_i) \approx |\det D\varphi_{\xi_i}|\,\text{area}(R_i).$

④ $$\int_V f(x)\,dx \approx \sum f(\text{some pt in } P_i)\,\text{area}(P_i)$$

$$= \sum f(\varphi(\xi_i)) \cdot |\operatorname{Det} D\varphi_{\xi_i}|\,\text{area}(R_i)$$

$$\approx \int_U f\circ\varphi(y)\,|\det D\varphi_y|\,dy \quad //.$$

⑤ Exactly same reasoning works in 3D.

because if $C_i \subseteq \mathbb{R}^3$ is a cuboid & $T: \mathbb{R}^3 \to \mathbb{R}^3$ is linear $\Big\}$ $\text{Volume}(TC_i) = |\det(T)|\,\text{vol}(C_i)$