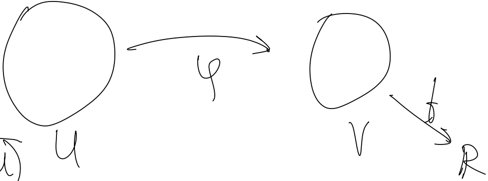
Coordinate Changes (Higher dim analog of "u-subst")

1 D: $\int_a^b f(x)\,dx \longrightarrow$ Put $x = g(y)$, $dx = g'(y)\,dy$

$x = a \iff y = g^{-1}(a)$, $x = b \iff y = g^{-1}(b)$

$$\int_{g^{-1}(a)}^{g^{-1}(b)} f(g(y))\, g'(y)\, du.$$

Higher dim: $U, V \subseteq \mathbb{R}^d$ ($d = 2$ or 3)



Assumptions: ① φ is a coordinate chg fn (φ is C^1, $\boxed{\text{bij}}$ & $D\varphi_x$ is inv $\forall x \in U$)

Thm: $$\int_V f(x)\,dx = \int_{y\in U} f(\varphi(y))\,|\det D\varphi_y|\,dy$$

$x = \varphi(y)$

$dx = |\det D\varphi_y|\,dy$

area integral of $U \subseteq \mathbb{R}^2$

Vol integral of $U \subseteq \mathbb{R}^3$

i.e. For area integrals: $$\int_V f\,dA = \int_U f\circ\varphi\,|\det D\varphi|\,dA$$

For volume integrals: $$\int_V f\,dV = \int_U f\circ\varphi\,|\det D\varphi|\,dV$$

Q1: Why? (IOU)

Q2: How do we use it?

① Higher D: Need φ to be $\boxed{\text{bij}}$

② Don't forget $|\operatorname{Det} D\varphi|$ (and not $\det D\varphi$).

Note: If φ fails to satisfy bij / $D\varphi$ is inv at finitely many points, then the theorem still works.

Also works if bij / $D\varphi$ is inv fail at finitely many "nice" curves

3D: " " " " " " " " " " " surfaces.

Eg: Compute the volume of a sphere of radius R. using coordinate changes.

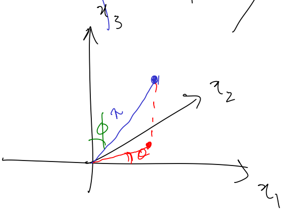
$$\mathrm{Vol} = \int_{B(0,R)} 1 \, \underbrace{dx}_{\substack{dx_1\, dx_2\, dx_3 \\ dV}}$$

Change to spherical coordinates: $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \varphi(r, \theta, \phi) = \begin{pmatrix} r \sin\phi \cos\theta \\ r \sin\phi \sin\theta \\ r \cos\phi \end{pmatrix}$

$$\Rightarrow \int_{B(0,R)} 1 \, dx_1\, dx_2\, dx_3 \overset{\text{spherical coord}}{=} \int 1 \, \underbrace{\left| \mathrm{Det}\, D\varphi \right|}_{r^2 \sin\phi} \, dr\, d\theta\, d\phi$$

$\{(r,\theta,\phi) \mid 0 \le r < R,\ \theta \in (-\pi,\pi),\ \phi \in (0,\pi)\}$

$$\overset{\text{Fubini}}{=} \int_{r=0}^{R} \int_{\theta=-\pi}^{\pi} \int_{\phi=0}^{\pi} 1 \, r^2 \sin\phi \, d\phi\, d\theta\, dr$$



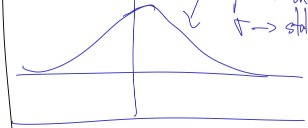
$$= \int_0^R 2\pi \cdot 2 \cdot r^2 \, dr = \frac{4\pi}{3} R^3$$ //

Eg 2: Compute $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}$ (1D int)

Bell curve: $\dfrac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma}$

$\mu \rightarrow$ mean

$\sigma \rightarrow$ std dev.



Trick: Let $I = \displaystyle\int_{-\infty}^{\infty} e^{-x^2} \, dx$

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} \, dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} \, dy\right)$$

$$= \int_{y=-\infty}^{\infty} \left(\int_{x=-\infty}^{\infty} e^{-x^2} \, dx\right) e^{-y^2} \, dy = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{\infty} e^{-(x^2+y^2)} \, dx \, dy$$

$$= \int_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dx\, dy \overset{\text{Polar Coordinates}}{=\!=} \int_{(\ r\ )} e^{-r^2}\, r\, dr\, d\theta$$

$$\{(r,\theta) \mid \theta \in (-\pi,\pi),\ r \in (0,\infty)\}$$

Fubini

$$= \int_{r=0}^{\infty} \int_{\theta=-\pi}^{\pi} e^{-r^2}\, r\, d\theta\, dr$$

$$x = r\cos\theta \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \varphi(r,\theta)$$

$$y = r\sin\theta$$

$$dx\,dy \longmapsto \underbrace{|\det D\varphi|}_{r}\, dr\, d\theta$$

Put $u = r^2$

$$= \int_{r=0}^{\infty} 2\pi\, e^{-r^2}\, r\, dr = \pi \int_0^{\infty} e^{-u}\, du$$

$$= \pi \ !$$

$$\Rightarrow I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \pi \Rightarrow \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \ !!$$