21-268: Multidimensional Calculus **Spring 2020**

Recitation March 31

Fubini's Theorem

Definition 1

$$S_xU = \{y|(x,y) \in U\} \qquad T_yU = \{x|(x,y) \in U\}$$

Theorem 2 *Let $U \subset \mathbb{R}^2$. Suppose $f : U \to \mathbb{R}$ satisfies*

$$\int_{x\in\mathbb{R}} \left(\int_{y\in S_xU} |f(x,y)|dy \right) dx < \infty$$

or

$$\int_{y\in\mathbb{R}} \left(\int_{x\in T_yU} |f(x,y)|dx \right) dy < \infty$$

the f is integrable over U and

$$\int_U f dA = \int_{x\in\mathbb{R}} \left(\int_{y\in S_xU} f(x,y)dy \right) dx = \int_{y\in\mathbb{R}} \left(\int_{x\in T_yU} f(x,y)dx \right) dy$$

Example 1

Compute the integral $\int_0^1 \int_y^1 e^{x^2} \, \mathrm{d}x \, \mathrm{d}y$.

Solution: We first check that

$$\int_0^1 \int_y^1 |e^{x^2}| \, \mathrm{d}x \, \mathrm{d}y \leq \int_0^1 \int_0^1 |e^{x^2}| \, \mathrm{d}x \, \mathrm{d}y \leq \int_0^1 \int_0^1 e \, \mathrm{d}x \, \mathrm{d}y = e < \infty$$

Thus, by Fubini's theorem,

$$\int_0^1 \int_y^1 e^{x^2} \, \mathrm{d}x \, \mathrm{d}y = \int_0^1 \int_0^x e^{x^2} \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 x e^{x^2} \, \mathrm{d}x = \frac{1}{2} e^{x^2} \bigg|_{x=0}^1 = \frac{e-1}{2}$$

Example 2

Use a triple integral to compute the volume of a cone with height h and base radius r.

Solution: Let the cone be $C = \{(x,y,z) \in \mathbb{R}^3 : z \in [0,h], \sqrt{x^2+y^2} \le \frac{r}{h}(h-z)\}$. Thus, after checking Fubini, we can express the volume of C as

$$\begin{aligned}
\int_C 1 \,\mathrm{d}V &= \int_0^h \int_{-r(h-z)/h}^{r(h-z)/h} \int_{-\sqrt{(r(h-z)/h)^2-y^2}}^{\sqrt{(r(h-z)/h)^2-y^2}} 1 \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z \\
&= \int_0^h \int_{-r(h-z)/h}^{r(h-z)/h} 2\sqrt{(r(h-z)/h)^2 - y^2} \,\mathrm{d}y \,\mathrm{d}z \\
&= \int_0^h \pi \left(\frac{r(h-z)}{h}\right)^2 \mathrm{d}z \\
&= \pi r^2 \int_0^h \left(1 - \frac{2z}{h} + \frac{z^2}{h^2}\right) \mathrm{d}z \\
&= \pi r^2 \left(h - h + \frac{1}{3}h\right) \\
&= \frac{1}{3}\pi r^2 h
\end{aligned}$$

What if we want to compute the mass of C given its density function $\rho(x,y,z)$? Then, the mass is

$$\int_C \rho \,\mathrm{d}V = \int_0^h \int_{-r(h-z)/h}^{r(h-z)/h} \int_{-\sqrt{(r(h-z)/h)^2-y^2}}^{\sqrt{(r(h-z)/h)^2-y^2}} \rho(x,y,z) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z$$

“counter example” of Fubini

Let $f : [0,1]^2 \to \mathbb{R}$ be defined via $f(x,y) = \dfrac{x^2-y^2}{(x^2+y^2)^2}$ First we check if the condition of Fubini is satisfies

$$\begin{aligned}
\int_0^1 \int_0^1 |f(x,y)| dy dx &\ge \int_0^1 \int_0^x |f(x,y)| dy dx \\
&= \int_0^1 \int_0^x \frac{x^2-y^2}{(x^2+y^2)^2} dy dx \\
&= \int_0^1 \frac{y}{x^2+y^2}\Big|_{y=0}^{x} dx \\
&= \int_0^1 \frac{1}{2x}\Big|_{y=0}^{x} dx \\
&= \frac{1}{2}\ln x\Big|_{x=0}^{1} = \infty
\end{aligned}$$

Similarly we have $\int_0^1 \int_0^1 |f(x,y)| dx dy = \infty$. So the condition for Fubini is not satisfied.

Compute the double integral in both ways

$$\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dx = \frac{-x}{x^2+y^2}\Big|_{x=0}^1 = \frac{-1}{1+y^2}$$

$$\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}dy = \frac{y}{x^2+y^2}\Big|_{y=0}^1 = \frac{1}{1+x^2}$$

$$\int_0^1\int_0^1 f(x,y)dxdy = \int_0^1 \frac{-1}{1+y^2}dy = -\arctan y|_{y=0}^1 = \frac{\pi}{4}$$

$$\int_0^1\int_0^1 f(x,y)dydx = \int_0^1 \frac{1}{1+x^2}dx = \arctan x|_{x=0}^1 = -\frac{\pi}{4}$$

They are not equal to each other

A preview of 21-325

Let $f:\mathbb{R}\to[0,\infty)$ define

$$\mathbf{P}[X>x] = \int_x^\infty f(y)dy$$

$$\mathbf{E}[X] = \int_0^\infty xf(x)dx$$

(for those who have seen some probability theory, these are the probability of $X>x$ and expectation for the continuous, non-negative variable X that has pdf f.)

We claim the following holds:

$$\mathbf{E}[X] = \int_0^\infty \mathbf{P}[X>x]dx$$

Proof: Let $\mathbb{1}_{y\leq x}:\mathbb{R}^2\to\{0,1\}$ be defined via $\mathbb{1}_{y\leq x}(x,y)=1$ when $y\leq x$ and $\mathbb{1}_{y\leq x}(x,y)=0$ when $y>x$ (we also call it the characteristic function of $y\leq x$). Then fix any x, $\int_0^\infty g(x,y)dy = x$ So

$$\mathbf{E}[X] = \int_0^\infty xf(x)dx = \int_0^\infty\int_0^\infty \mathbb{1}_{y\leq x}(x,y)dyf(x)dx = \int_0^\infty\int_0^\infty \mathbb{1}_{y\leq x}(x,y)f(x)dydx$$

Assume Fubini holds,

$$\mathbf{E}[X] = \int_0^\infty\int_0^\infty g(x,y)f(x)dxdy = \int_0^\infty\int_y^\infty 1f(x)dxdy = \int_0^\infty \mathbf{P}[X>y]dy$$

Another equivalent way of doing it without using the characteristic function is to observe

$$xf(x) = \int_0^x f(x)dy$$

So

$$\mathbf{E}[X] = \int_0^\infty xf(x)dx = \int_0^\infty \int_0^x f(x)dydx = \int_0^\infty \int_y^\infty f(x)dxdy = \int_0^\infty \mathbf{P}[X > y]dy$$

Now if Fubini does not hold, then both

$$\int_0^\infty \int_0^\infty g(x,y)f(x)dxdy = \infty$$

$$\int_0^\infty \int_0^\infty g(x,y)f(x)dydx = \infty$$

So they are still the same. (In general same logic follows for all non-negative functions) ∎