Last time: triple integrals.

$U \subseteq \mathbb{R}^3$ nice domain.

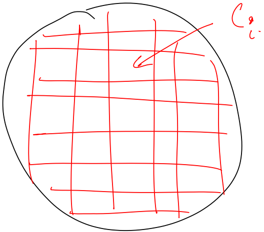
$\rho : U \to \mathbb{R}$ a nice fn.

Intuition: $U \to$ region with uneven density

$\rho(x)$ = density at $x \in U$.

Q: Total mass of U. $\longrightarrow$ Triple integral.

Notation: $\displaystyle \int_U \rho \underbrace{dV}_{\text{volume int}} = \iiint_U \rho \, dV = \int_U \rho(x)\, dx$



U

(1) Divide U into many small cubes C_i (ignore cubes that are not compl cont in U).

(2) For each cube C_i, pick $\xi_i \in C_i$ (e.g. $\xi_i =$ center C_i)

(3) Approx mass of $U = \sum$ mass of each cube

$$\approx \sum \mathrm{vol}(C_i) \cdot \rho(\xi_i)$$

(4) $$\int_U \rho \, dV \overset{\text{def}}{=} \lim_{\|P\| \to 0} \sum \mathrm{vol}(C_i)\, \rho(\xi_i).$$

($\|P\|$ = mesh size of the partition = largest of the lengths of each side of each C_i)

Fubini works exactly the same way in 3D.

(if one iterated integral $|f|$ is finite then all iterated integrals (any order) are equal)

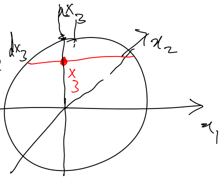
① Eg: Compute the volume of a sphere of radius R.
i.e. Let $\mathcal{U} = \{ x \in \mathbb{R}^3 \mid |x| < R \}$. Compute $\underbrace{\int_{\mathcal{U}} 1 \, dV}$ ← triple integral.

Let $V = \int_{B(0,R)} 1 \, dV$ = ~~write~~ write this as an 3 iterated integrals & evaluate.

$$= \int_{x_3=-R}^{+R} \int_{x_2=-\sqrt{R^2-x_3^2}}^{+\sqrt{R^2-x_3^2}} \int_{x_1=-\sqrt{R^2-x_2^2-x_3^2}}^{+\sqrt{R^2-x_2^2-x_3^2}} 1 \, dx_1 \, dx_2 \, dx_3$$

goes here

goes here



Given $x_3 \in \mathbb{R}$, what is the min & max values x_2 can take if $(x_1, x_2, x_3) \in \mathcal{U}$?

$x \in U \Rightarrow |x| < R \Rightarrow x_1^2 + x_2^2 + x_3^2 \leq R \iff x_2^2 \leq R^2 - x_1^2 - x_3^2$

min/max obtained when $x_1 = 0$.

$x_2 = +\sqrt{R^2 - x_3^2}$ (max)

$x_2 = -\sqrt{R^2 - x_3^2}$ (min)

Bounds for x_1: Given (x_2, x_3) what is the min & max x_1 can take if $(x_1, x_2, x_3) \in U$

min → lower bd

max → upper bd.

$$\Rightarrow \int_U 1 \, dV = \int_{x_3=-R}^{R} \int_{x_2=-\sqrt{R^2-x_3^2}}^{+\sqrt{R^2-x_3^2}} \int_{x_1=-\sqrt{R^2-x_3^2-x_2^2}}^{+\sqrt{R^2-x_3^2-x_2^2}} 1 \, dx_1 \, dx_2 \, dx_3$$

("The fn 1 is +ve. If final ans is $< \infty$ $\Rightarrow$ Fubini applies")

$$= 2 \int_{-R}^{R} \int_{x_2=-\sqrt{R^2-x_3^2}}^{+\sqrt{R^2-x_3^2}} \sqrt{R^2-x_3^2-x_2^2} \, dx_2 \, dx_3$$

Substitute

$$x_2 = \underbrace{\sqrt{R^2-x_3^2}}_{r} \sin\theta$$

$$dx_2 = r\cos\theta \, d\theta$$

$$\sqrt{r^2 - r^2\sin^2\theta} = r\cos\theta$$

$$= 2 \int_{x_3=-R}^{R} \int_{\theta=-\pi/2}^{\pi/2} r\cos\theta \; r\cos\theta \, d\theta \, dx_3$$

$$= 2\int_{x_3=-R}^{R}\int_{\theta=-\pi/2}^{\pi/2}(R^2-x_3^2)\cos^2\theta\,d\theta\,dx_3$$

$$= 2\int_{x_3=-R}^{R}(R^2-x_3^2)\underbrace{\int_{\theta=-\pi/2}^{\pi/2}\cos^2\theta\,d\theta}\,dx_3$$

$$= \pi\int_{x_3=-R}^{R}(R^2-x_3^2)\,dx_3 = 2\pi\left[R^2x_3-\frac{x_3^3}{3}\right]_0^R$$

$$= 2\pi\left(\frac{2R^3}{3}\right) = \boxed{\frac{4\pi R^3}{3}}$$

$\sin^2\theta+\cos^2\theta=1$

$\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta = \int_{-\pi/2}^{\pi/2}\sin^2\,d\theta$ (symm).

$\Rightarrow \int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta = \frac{1}{2}\int_{-\pi/2}^{\pi/2}1$

$\boxed{=\pi/2}$

Coordinate changes: 1D $\longrightarrow$ "u - substitution".

1D: $\displaystyle\int_a^b f(x)\,dx. \longrightarrow$ Put $x = g(u)$

$\Downarrow$

$dx = g'(u)\,du.$

$x = a \iff u = g^{-1}(a)$

$x = b \iff u = g^{-1}(b)$

$$\int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))\,g'(u)\,du.$$

$(x \in \mathbb{R}^2$. Put $x = \varphi(u)$; φ

$(\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2)$

Q: Higher dim:

$$\int_U f(x)\,\cancel{dA} \longrightarrow \int_U f(\varphi(u))\,\cancel{dx}\ D\varphi\ du$$

dA / dx / area int

$|\det D\varphi|$

(& need bij (next time!))