

hard time: triple integrals.

$U \subseteq \mathbb{R}^3$  nice domain.

$f: U \rightarrow \mathbb{R}$  a nice fn.

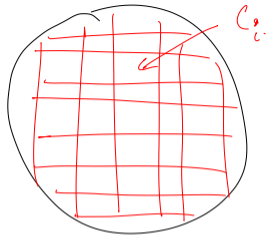
Situation:  $U \rightarrow$  region with uneven density

$f(x) =$  density at  $x \in U$ .

Q: Total mass of  $U$ .  $\rightarrow$  Triple integral.

$$\text{Notation: } \int_U f \, dV = \iiint_U f \, dV = \int_U f(x) \, dx$$

you  
volume  
int



$U$

① Divide  $U$  into many small cubes ( $C_i$  ignore cubes that are not compl cont in  $U$ ).

(2) For each cube  $C_i$ , pick  $\xi_i \in C_i$  (e.g.  $\xi_i = \text{center } C_i$ )

(3) Approx mass of  $U = \sum \text{mass of each cube}$   
 $\approx \sum \text{vol}(C_i) \cdot \rho(\xi_i)$

(4)  $\int_U \rho \, dV \stackrel{\text{def}}{=} \lim_{\|P\| \rightarrow 0} \sum \text{vol}(C_i) \rho(\xi_i)$ .

$\|P\| = \text{mesh size of the partition}$   
 $= \text{largest of the lengths of each side of each } C_i$

Fubini works exactly the same way in 3D.  
(if one iterated integral  $\|f\|$  is finite then all iterated integrals (any order) are equal)

① Eg: Compute the volume of a sphere of radius  $R$ .  
 i.e. Let  $U = \{x \in \mathbb{R}^3 \mid |x| < R\}$ . Compute  $\int_U 1 \, dV$   
 triple integral.

Let  $V = \int_{B(0,R)} 1 \, dV =$  write this as an iterated integrals & evaluate.

$$\int_{-R}^R \int_{-\sqrt{R^2-x_3^2}}^{\sqrt{R^2-x_3^2}} \int_{-\sqrt{R^2-x_2^2-x_3^2}}^{\sqrt{R^2-x_2^2-x_3^2}} 1 \, dx_1 \, dx_2 \, dx_3$$

Diagram illustrating the iterated integral for the volume of a sphere. The sphere is shown in the  $x_1, x_2, x_3$  coordinate system. The limits of integration are indicated by the nested integrals above the diagram. The innermost integral is over  $x_1$ , the middle over  $x_2$ , and the outermost over  $x_3$ . The limits for  $x_2$  are  $-\sqrt{R^2-x_3^2}$  and  $\sqrt{R^2-x_3^2}$ . The limits for  $x_1$  are  $-\sqrt{R^2-x_2^2-x_3^2}$  and  $\sqrt{R^2-x_2^2-x_3^2}$ . The limits for  $x_3$  are  $-R$  and  $R$ . A red dot on the  $x_3$  axis is labeled  $x_3$ . A red arrow points from the text "Given  $x_3 \in \mathbb{R}$ , what is the min & max values  $x_2$  can take if  $(x_1, x_2, x_3) \in U$ ?" to the  $x_2$  limits in the integral. A green arrow points from the text "goes here" to the  $x_3$  limits in the integral.

Given  $x_3 \in \mathbb{R}$ , what is the min & max values  $x_2$  can take if  $(x_1, x_2, x_3) \in U$ ?

$$x \in U \Rightarrow |x| < R \Rightarrow x_1^2 + x_2^2 + x_3^2 \leq R^2 \Leftrightarrow x_2^2 \leq R^2 - x_1^2 - x_3^2$$

min/max obtained when  $x_1 = 0$ .

$$x_2 = +\sqrt{R^2 - x_3^2} \quad (\text{max})$$

$$x_2 = -\sqrt{R^2 - x_3^2} \quad (\text{min}).$$

Bounds for  $x_1$ : Given  $(x_2, x_3)$  what is the min & max  $x_1$  can take  
 if  $(x_1, x_2, x_3) \in U$

$\downarrow$  lower bd       $\downarrow$  upper bd.

$$\Rightarrow \int_{\Omega} 1 \, dV = \int_{x_3 = -R}^R \int_{x_2 = -\sqrt{R^2 - x_3^2}}^{+\sqrt{R^2 - x_3^2}} \int_{x_1 = -\sqrt{R^2 - x_3^2 - x_2^2}}^{+\sqrt{R^2 - x_3^2 - x_2^2}} 1 \, dx_1 \, dx_2 \, dx_3$$

$$= 2 \int_{-R}^R \int_{x_2 = -\sqrt{R^2 - x_3^2}}^{+\sqrt{R^2 - x_3^2}} \sqrt{R^2 - x_3^2 - x_2^2} \, dx_2 \, dx_3$$

$$= 2 \int_{x_3 = -R}^R \int_{\theta = -\pi/2}^{\pi/2} r \cos \theta \cdot r \cos \theta \, d\theta \, dx_3$$

$$dx_2 \, dx_3$$

(The fun 1 is +ve.  
If final ans is <math>\infty</math>  
 $\Rightarrow$  Fubini applies)

Substitute  $x_2 = \sqrt{R^2 - x_3^2} \sin \theta$

$$dx_2 = r \cos \theta \, d\theta$$

$$\sqrt{r^2 - r^2 \sin^2 \theta} = r \cos \theta$$

$$= 2 \int_{x_3 = -R}^R \int_{\theta = -\pi/2}^{\pi/2} (R^2 - x_3^2) \cos^2 \theta \, d\theta \, dx_3$$

$$= 2 \int_{x_3 = -R}^R (R^2 - x_3^2) \int_{\theta = -\pi/2}^{\pi/2} \cos^2 \theta \, d\theta \, dx_3$$

$$= \pi \int_{x_3 = -R}^R (R^2 - x_3^2) \, dx_3 = 2\pi \left[ R x_3 - \frac{x_3^3}{3} \right]_{-R}^R$$

$$= 2\pi \left( \frac{2R^3}{3} \right) = \boxed{\frac{4\pi R^3}{3}}$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\int_{-\pi/2}^{\pi/2} \cos^2 \theta \, d\theta =$$

$$= \int_{-\pi/2}^{\pi/2} \sin^2 \theta \, d\theta \quad (\text{symmetry})$$

$$\int_{-\pi/2}^{\pi/2} \cos^2 \theta \, d\theta =$$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} \cos^2 \theta \, d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} 1 \, d\theta$$

$$= \boxed{\frac{\pi}{2}}$$

Coordinate changes:  $1D \rightarrow$  "u-substitution".

1D:  $\int_a^b f(x) dx \rightarrow$  Put  $x = g(u)$

$dx = g'(u) du$

$x = a \Leftrightarrow u = g^{-1}(a)$

$x = b \Leftrightarrow u = g^{-1}(b)$

$\int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u)) g'(u) du$

$x \in \mathbb{R}^2$ . Put  $x = \varphi(u)$ :  $\varphi$   
( $\varphi: \mathbb{R} \rightarrow \mathbb{R}^2$ )

Q: Higher dim:

$\int_U f(x) dA$   
*area int*

$\rightarrow \int_U f(\varphi(u))$

$\circlearrowleft D\varphi \circlearrowright du$

$|\det D\varphi|$   
(& need  $bij$ )  
(next time!)