Integration: $U \subseteq \mathbb{R}^2$

$P \to$ "partition" of U into rectangles R_{ij}

(ignore rectangles that are not completely contained in U).

$\|P\| \to$ norm $P \to$ mesh size of P

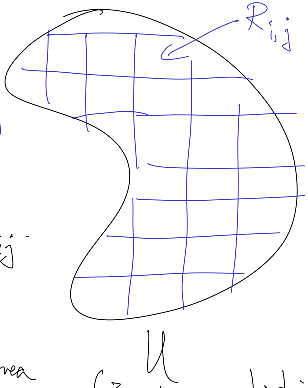
= largest side length of all R_{ij}.

$f : U \to \mathbb{R}$.

<u>**Def:**</u> $$\int_U f \, dx_1 \, dx_2 = \int_U f \, \underbrace{dA}_{\text{area}}$$

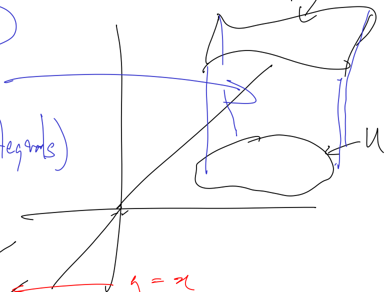
$$= \lim_{\|P\| \to 0} \sum_{i,j} \text{area}(R_{ij}) \cdot f(\xi_{ij})$$

(ξ_{ij} is any point in $R_{i,j}$)



Intuition: $\int_U f \, dA$ = volume under the surface graph of f

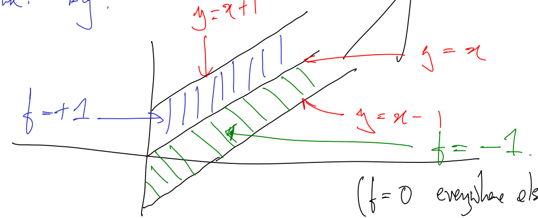
Volume =



Goal 1: Reduce double integrals to iterated integrals (2 single integrals)

Catch: doesn't always work: Eg.

$U = \mathbb{R}^2$.



($f = 0$ everywhere else).

Formula: $f(x,y) = \begin{cases} +1 & x \le y < x+1 \ \& \ x > 0 \\ -1 & x-1 \le y < x \ \& \ x > 0 \end{cases}$

Compute $\int_{\mathbb{R}^2} f \, dA$.

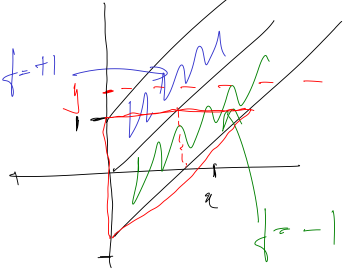
"Iterated Integrals"

$$\int\left(\int_{S_x} f(x,y)\,dy\right)dx$$

Guess: $\int_{\mathbb{R}^2} f(x,y)\, dA = \int_{x\in\mathbb{R}} \left(\int_{y\in\mathbb{R}} f(x,y)\, dy \right) dx \leftarrow 1$

Second Guess: $\int_{\mathbb{R}^2} f(x,y)\, dA = \int_{y\in\mathbb{R}} \left(\int_{x\in\mathbb{R}} f(x,y)\, dx \right) dy \leftarrow 2$

Compute ①:
$$\int_{x\in\mathbb{R}} \left(\int_{y\in\mathbb{R}} f(x,y)\,dy \right) dx = \int_{x\in\mathbb{R}} (0)\,dx = 0$$



Compute ②:
$$\int_{y\in\mathbb{R}} \left(\int_{x\in\mathbb{R}} f(x,y)\,dx \right) dy = \underbrace{\int_{y=-\infty}^{1} \left(\int_{x\in\mathbb{R}} f(x,y)\,dx \right) dy}_{\text{compute with formula } = 1} + \overbrace{\int_{y=1}^{\infty} \left(\int_{x\in\mathbb{R}} f(x,y)\,dx \right) dy}^{0}$$

i.e. In this Eg:

$$\int_{x\in\mathbb{R}} \left(\int_{y\in\mathbb{R}} f(x,y)\,dy \right) dx = 0$$

$$\neq -1 = \int_{y\in\mathbb{R}} \left(\int_{x\in\mathbb{R}} f(x,y)\,dx \right) dy.$$

(Reminiscent of "conditional convergence of series")

Recall the series $\sum\limits_{1}^{\infty} a_i$ is abs cgt if $\sum\limits_{i=1}^{\infty} |a_i| < \infty$

" " $\sum a_i$ is cond cgt if ① $\sum a_i$ is cgt & ② $\sum |a_i| = +\infty$

Didn't work because $\int_{\mathbb{R}^2} |f(x,y)|\, dA = +\infty.$

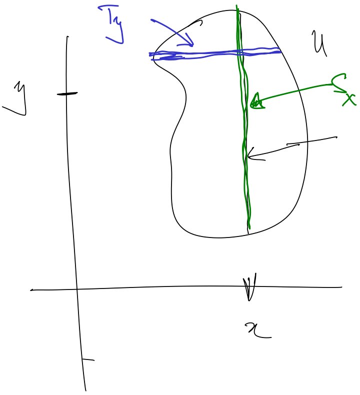
Fubini's theorem: $U \subseteq \mathbb{R}^2$.

S_x = Vertical slice of U through x

$= \{ y \in \mathbb{R} \mid (x,y) \in U \}$

T_y = Horizontal slice of U through y

$= \{ x \in \mathbb{R} \mid (x,y) \in U \}$.



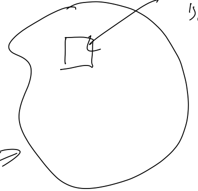
Theorem (Fubini): If *either* (1) $\displaystyle\int_{y\in\mathbb{R}} \left(\int_{x\in T_y} |f(x,y)|\, dx \right) dy < \infty$

OR (2) $\displaystyle\int_{x\in\mathbb{R}} \left(\int_{y\in S_x} |f(x,y)|\, dy \right) dx < \infty$

Then
$$\int_U f\, dA = \int_{x\in\mathbb{R}} \left(\int_{y\in S_x} f(x,y)\, dy \right) dx$$
$$= \int_{y\in\mathbb{R}} \left(\int_{x\in T_y} f(x,y)\, dx \right) dy$$

Same works in $\mathbb{R}^3$.

① Integrals in $\mathbb{R}^3$:



Q: Mass of the ball $\leftarrow$ Ball in $\mathbb{R}^3$ of varying density

Divide Ball into cubes: $C_{i,j,k}$

Pick $\xi_{i,j,k} \in C_{ijk}$

$$\underset{U}{\int} \rho \underbrace{dV}_{\text{volume integral}} = \lim_{\|P\| \to} \underbrace{\sum \text{vol}(C_{ijk}) \cdot \rho(\xi_{ijk})}$$

$\rho(\xi_{ijk}) =$ density at this point.