21-268: Multidimensional Calculus **Spring 2020**

Recitation March 24

Tomorrow we have some very specific event...

Problem Let $g : \mathbb{R} \to \mathbb{R}$ be defined via $g(x) = f(x, x^2)$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function. Prove that g is differentiable and $g'(a) = \partial_1 f(a, a^2) + 2a\partial_2 f(a, a^2)$ for any $a \in \mathbb{R}$ (Of course, you cannot use the chain rule to prove this statement)

Solution This is a special (and very simplified) case of the multidimensional chain rule $D(f \circ g)_a = Df_{g(a)} Dg_a$.

Let $a \in \mathbb{R}$, by definition of the derivative,

$$\lim_{(h_1,h_2)\to(0,0)} \frac{|f(a + h_1, a^2 + h_2) - f(a, a^2) - \partial_1 f(a, a^2)h_1 - \partial_2 f(a, a^2)h_2|}{|(h_1, h_2)|} = 0$$

Substitute in $h_2 = 2ah_1 + h_1^2$ (it is a continuous function with respect to h_1!) we get

$$\lim_{h\to 0} \frac{|f(a + h, (a + h)^2) - f(a, a^2) - \partial_1 f(a, a^2)h - \partial_2 f(a, a^2)(2ah + h^2)|}{|(h, 2ah + h^2)|} = 0$$

$|(h, 2ah + h^2)| \leq (2|a| + 2)h$ when $h < 1$, so

$$\lim_{h\to 0} \frac{|f(a + h, (a + h)^2) - f(a, a^2) - \partial_1 f(a, a^2)h - \partial_2 f(a, a^2)(2ah + h^2)|}{(2|a| + 2)|h|} = 0$$

$\lim_{h\to 0} \partial_2 f h^2/|h| = 0$ so add it in and multiply by $2|a| + 2$ we get

$$\lim_{h\to 0} \frac{|f(a + h, (a + h)^2) - f(a, a^2) - (\partial_1 f(a, a^2) + 2a\partial_2 f(a, a^2))h|}{|h|} = 0$$

which by definition means $g'(a) = \partial_1 f(a, a^2) + 2a\partial_2 f(a, a^2)$

Problem

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function in C^2. Assume for a point $a \in \mathbb{R}$, $Df_a = 0$ and $Hf_a = -I$. Prove that f achieves a local maximum at a.

Solution

This is simplified case of the theorem we have for general negative definite Hf_a.

By Taylor's theroem we know

$$f(x) = f(a) + Df_a(x-a) + \frac{1}{2}(x-a)^T Hf_a(x-a) + R_2(x-a)$$

for some function $R_2(x-a)$ and $\lim_{x\to a} \dfrac{R_2(x-a)}{|x-a|^2} = 0$. Substitute in $Df_a = 0, Hf_a = -I$ we get

$$f(x) = f(a) - \frac{1}{2}|x-a|^2 + R_2(x-a)$$

Since $\lim_{x\to a} \dfrac{R_2(x-a)}{|x-a|^2} = 0$,

$$\exists r > 0, |x-a| < r \Rightarrow \frac{|R_2(x-a)|}{|x-a|^2} < 1/2 \Rightarrow |R_2(x-a)| < \frac{1}{2}|x-a|^2$$

Then for all $x \in B(a,r)$

$$f(x) = f(a) - \frac{1}{2}|x-a|^2 + R_2(x-a) <= f(a)$$

So f attains a local maximum at a.

Problem

Let $f(x,y) = Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F$ where A, B, C, D, E, F are real numebrs. Assume $A > 0$ and $B^2 < AC$. Prove that $f(x,y)$ has a local minimum.

Solution

First compute

$$Df_{(x,y)} = \begin{bmatrix} 2Ax + 2By + 2D & 2Cy + 2Bx + 2E \end{bmatrix} = 2\begin{bmatrix} A & B \\ B & C \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + 2\begin{bmatrix} D \\ E \end{bmatrix}$$

$$Hf_{(x,y)} = 2\begin{bmatrix} A & B \\ B & C \end{bmatrix}$$

Since $A > 0$ and $AC - B^2 > 0$, recall Sylvester's law of signs, Hf is positive definite. Also $\begin{bmatrix} A & B \\ B & C \end{bmatrix}$ is invertible so $Df_{(x_1,y_1)} = 0$ when

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} A & B \\ B & C \end{bmatrix}^{-1}\begin{bmatrix} D \\ E \end{bmatrix}$$

So we can find a point $a = (x_1, y_1)$ such that $Df_a = 0$ and Hf_a is positive definite, thus it is a local minimum.

Problem

Show that there exists an open set $U \subseteq \mathbb{R}$ and two C^1 functions $f, g : U \to \mathbb{R}$ such that for all $x \in U$,

$$1 \in U, \quad f(1) = 1, \quad g(1) = 0, \quad \text{and} \quad \begin{cases} e^{xf(x)g(x)} + xf(x)g(x) = 1 \\ xf(x)^3 g(x) + f(x) = 1 \end{cases} \quad \text{for all } x \in U$$

Solution Consider the function $F : \mathbb{R}^3 \to \mathbb{R}^2$ which is defined as

$$F(x, y, z) = \begin{pmatrix} F_1(x, y, z) \\ F_2(x, y, z) \end{pmatrix} = \begin{pmatrix} e^{xyz} + xyz \\ xy^3 z + y \end{pmatrix}$$

Note that $F(1, 1, 0) = (1, 1)$, and

$$DF_{(x,y,z)} = \begin{bmatrix} yze^{xyz} + yz & xze^{xyz} + xz & xye^{xyz} + xy \\ y^3 z & 3xy^2 z + 1 & xy^3 \end{bmatrix}, \quad DF_{(1,1,0)} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

Since $\det \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \neq 0$, by implicit function theorem, there exists an open set $U \subseteq \mathbb{R}$ and a C^1 function $h : U \to \mathbb{R}^2$ such that

$$h(1) = (1, 0), \quad \text{and} \quad F(x, h(x)) = (1, 1) \text{ for all } x \in U$$

We can then let $f = h_1$ and $g = h_2$ so we can get the desired result.

Problem

In the region $x, y, z \geq 0$, maximize xyz^2 subject to the constraint $x + y + z = C$ for some $C > 0$.

Solution

The constraint $x+y+z = C$ infers that $(x, y, z) \in [0, C]^3$, which is a bounded set. This means that the constraint global maximum of xyz^2 either occurs as a constraint local maximum in the interior $(0, C)^3$, or occurs on the boundary of $[0, C]^3$.

Let $f(x, y, z) = xyz^2$ and $g(x, y, z) = x + y + z$ for $x, y, z \geq 0$. We first check the interior region $(x, y, z) \in (0, C)^3$. By solving $g = C$ and $\nabla f = \lambda \nabla g$ simultaneously, we get

$$x + y + z = C, \qquad \begin{pmatrix} yz^2 \\ xz^2 \\ 2xyz \end{pmatrix} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

This leads to $x = y = \frac{C}{4}$ and $z = \frac{C}{2}$. Thus, there is only one candidate in the interior: $f(\frac{C}{4}, \frac{C}{4}, \frac{C}{2}) = \frac{C^4}{64}$.

We then check the boundary, i.e., one of x, y, z is 0 or C. If $x = 0$, then $f(x, y, z) = 0 < \frac{C^4}{64}$. If $x = C$, then since $x + y + z = C$ and $x, y, z \geq 0$, we must have $y = z = 0$, so that $f(x, y, z) = 0 < \frac{C^4}{64}$. The same thing happens if $y = 0$ or $y = C$ or $z = 0$ or $z = C$.

Therefore, the maximum of xyz^2 subject to the constraint $x + y + z = C$ is $\frac{C^4}{64}$.