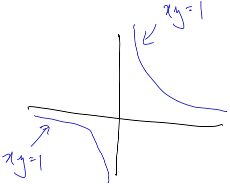
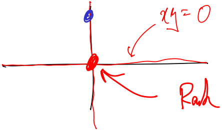
$f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad$ Rank $(Df_a) = 0$

Eg: $f(x,y) = x \cdot y \longrightarrow Df_{(x,y)} = \begin{pmatrix} y & x \end{pmatrix}$

$\{f = c\}$ $\quad c \neq 0$

If $c \neq 0$, $\forall (x,y) \in \{f = c\}$, Rank $Df_{(x,y)} = 1$





$xy = 0$

Rank $(Df_{(0,0)}) = 0 < 1.$

Q: Assume: $f: \mathbb{R}^d \longrightarrow \mathbb{R}$ is C^2.
$\nabla f(a) = 0$, Hf_a is +ve def.

Claim: f has a local min at a.

~~Pf:~~ Scratch: Taylor

dxd matrix

$$f(x) = f(a) + \nabla f(a) \cdot (x-a) + \frac{1}{2}(x-a) \cdot Hf_a (x-a) + R_2(x)$$

Know $\lim_{x \to a} \frac{R_2(x)}{(x-a)^2} = 0$

at a:)

At a : $$f(x) = f(a) + \underbrace{\nabla f(a)\cdot(x-a)}_{0} + \underbrace{\frac{1}{2}(x-a)\cdot\left(Hf_a(x-a)\right)}_{\geq \frac{\lambda_0}{2}|x-a|^2} + \underbrace{R_2(x)}_{\text{smaller than } |x-a|^2}$$

Want $f(x) > f(a)$ for x close to a.

$\lambda_0 > 0$ ($\because Hf_a$ is +ve def)

Need to Ensure $$\frac{\lambda_0}{2}|x-a|^2 + R_2(x) > 0$$

Choose $\varepsilon = \frac{\lambda_0}{2}$: $|x-a| < \delta \Rightarrow \left|\frac{R_2(x)}{|x-a|^2}\right| < \varepsilon = \frac{\lambda_0}{2}$.

$$\Rightarrow R_2(x) \in \left(-\frac{\lambda_0}{2}|x-a|^2, +\frac{\lambda_2}{2}|x-a|^2\right)$$

$$\Rightarrow \frac{\lambda_0}{2}|x-a|^2 + R_2(x) > 0$$

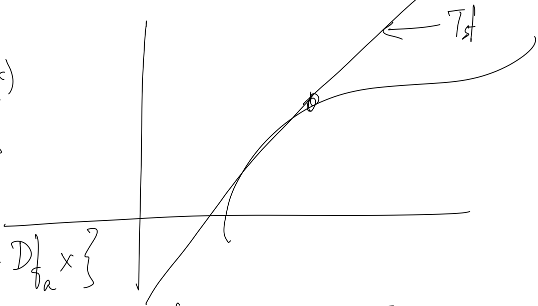
QED

$y = f(x)$

$\Gamma = \text{Graph of } \{y = f(x)\}$

$= \{(x,y) \mid y = f(x)\}$

Tgt space at a: $\{(x,y) \mid y = Df_a\, x\}$



Tgt plane at a: $\{(x,y) \mid y - f(a) = Df_a (x-a) \}$

pass thg $(a, f(a))$

$\left(\quad y = f(a) + \cancel{f'}(a)\,(x-a) \right.$

Df_a

Say $f: \mathbb{R}^{d+n} \longrightarrow \mathbb{R}^n$. $M = \{f = c\}$ $(\text{Rank } Df_a = n \ \forall a \in M)$.

Q: $a \in M \longrightarrow$ Tgt space of M at a: $\underbrace{\ker(Df_a)}$

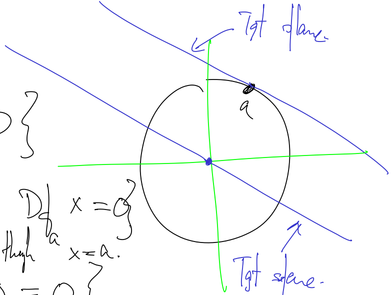
Q: What is the Tgt "hyper" plane?

Never: $\{x \in \mathbb{R}^{d+n} \mid Df_a(x-a) = 0\}$

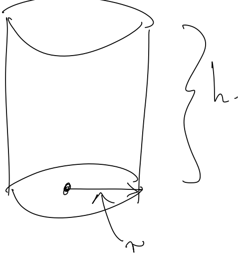
Know Tgt space $= \ker(Df_a) = \{x \in \mathbb{R}^{d+n} \mid Df_a x = 0\}$

Tgt plane $=$ shift $\nearrow$ so it passes through $x = a$.

$= \{x \in \mathbb{R}^{d+n} \mid Df_a(x-a) = 0\}$.



Eg: Max vol of cyl
given surface area $= 6\pi a^2$



$V(r,h) = \pi r^2 h$ = volume.
$S(r,h) = 2\pi r h + 2\pi r^2$

Goal: max V given $S = 6\pi a^2$

Lag Mult: At a const min/max: $\boxed{\nabla V = \lambda \nabla S}$

Gen Lag mult: max/min f given $g = c$ $(g : \mathbb{R}^{d+n} \to \mathbb{R}^n)$

Constr max/min $\longrightarrow$ $\nabla f(a) = \sum_{i=1}^{m} \lambda_i \nabla g_i(a)$ $\quad (\& \ g(a) = c)$

Our case: $f = V$, $g : \mathbb{R}^2 \to \mathbb{R}$ $\quad (g(r,h) = S(r,h))$

At constr min/max $\boxed{\nabla V = \lambda \nabla S}$

$V(r,h) = \pi r^2 h.$

$S(r,h) = 2\pi r h + 2\pi r^2.$

$$\nabla V = \begin{pmatrix} 2\pi r h \\ \pi r^2 \end{pmatrix} \qquad \nabla S = \begin{pmatrix} 2\pi h + 4\pi r \\ 2\pi r \end{pmatrix}$$

Solve $\nabla V = \lambda \nabla S \iff$

$$\cancel{2\pi} r h = \lambda (\cancel{2\pi} h + \overset{2}{\cancel{4}}\cancel{\pi} r)$$

$$\cancel{\pi} \cancel{r^2} = \lambda \, 2\cancel{\pi}\cancel{r} \quad \longleftarrow \quad r = 2\lambda$$

$$\cancel{2}\pi r h + \cancel{2}\pi r^2 = \overset{3}{\cancel{6}} \cancel{\pi} a^2$$

$\lambda = \frac{r}{2}$: $\quad \cancel{r}h = \frac{\cancel{r}}{2}(h + 2r) \Rightarrow h = 2r$

$$\Rightarrow r(2r) + r^2 = 3a^2 \Rightarrow r = \pm a.$$

$r = +a \longrightarrow$ f constr local max.

Max Val : $\pi r^2 h = \pi a^2 (2a) = \boxed{2\pi a^3}$

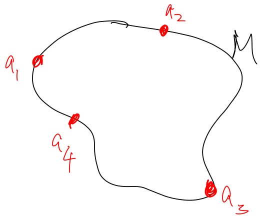
General principle : Say the constr $M = \{g = c\}$ is closed & bdd.

Then (analysis) $\Rightarrow$ f necessarily attains a max & min on M.

Let $a_1, \dots a_m$ be all points at which the Lagrange multiplier eqns have a sol. i.e. $\forall a_j, \exists \lambda_1, \dots \lambda_n \ni \nabla f(a_j) = \sum_{i=1}^{n} \lambda_i \nabla g_i(a_j)$

Then the point at which $f(a_i)$ is the largest **MUST** be the constr global max

& The point at which $f(a_i)$ is smallest must be the constr global min.

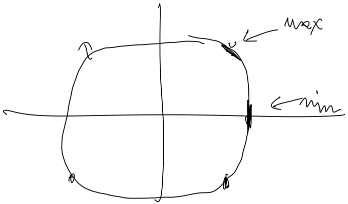


$$T = \left\{(x,y) \,\middle|\, \cos x \cos y = \frac{1}{2}, \ x \in (-\pi/2, \pi/2)\right\} = \{ g(x,y) = c \}$$

Goal: max & min K.

max/min K given the constr $\cos x \cos y = \frac{1}{2}$

$$K = \frac{\left|\partial_x^2 g \, (\partial_y g)^2 + (\partial_y^2 g)(\partial_x g)^2 - 2\partial_x \partial_y g \, \partial_x g \, \partial_y g\right|}{|\nabla g|^3}$$


max
min