Let $U = \{ (x,y) \in \mathbb{R}^2 \mid x+y < y^2 \}$ NTS $U \to$ is open.

Idea: Pick $(a,b) \in U$. NTF $\delta > 0$ + $|(a,b) - (x,y)| < \delta$

$\Rightarrow (x,y) \in U$.

Note: $(a,b) \in U \Rightarrow a+b-b^2 < 0 \Leftrightarrow \underbrace{b^2 - (a+b)}_{c} > 0$.

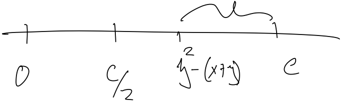
Q: If $|(x,y) - (a,b)| < \delta$ (chosen δ small).

can you make Make $\to$ $\underbrace{|y^2 - (x+y) - (\underbrace{b^2 - (a+b)}_{c})|}_{< \frac{c}{2}}$ super small?

(Note If $\left| y^2 - (x+y) - c \right| < \frac{c}{2} \Rightarrow y^2 - (x+y) > \frac{c}{2}$.

$\Rightarrow y^2 - (x+y) > 0$

$\Rightarrow (x,y) \in U$.



$x \in \mathbb{R} \longrightarrow |x| =$ abs value of x

$x \in \mathbb{R}^n \longrightarrow x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad |x| = \sqrt{x_1^2 + \cdots x_n^2}$

$(\|x\|)$

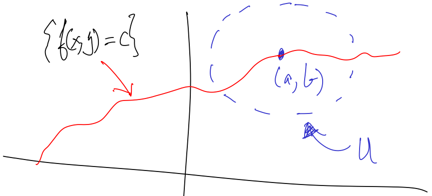
Implicit fn thm: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ f is C^1

$f(a,b) = c \in \mathbb{R}$. Assume $\partial_y f(a,b) \neq 0$

NTS: $\exists\, U$ open $\ni$

$$\{(x,y) \mid (x,y) \in U \ \&\ f(x,y) = c\}$$

$$= \{(x,y) \mid y = \underbrace{g(x)}_{\exists g,\ C'} \ \&\ x \in \underbrace{U'}_{\exists U',\ \text{open.}}\}$$



$$Df_{(a,b)} = \begin{pmatrix} \underbrace{\partial_x f(a,b)}_{*} & \underbrace{\partial_y f(a,b)}_{\neq 0} \end{pmatrix}$$

__Proof:__ Idea 1: (Thin air)

Let $\varphi(x,y) = \begin{pmatrix} x \\ f(x,y) \end{pmatrix}$

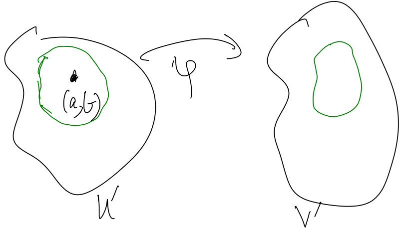
New coordinates: $\begin{pmatrix} u \\ v \end{pmatrix} = \varphi(x,y) = \begin{pmatrix} x \\ f(x,y) \end{pmatrix}$ $\quad (\varphi: \mathbb{R}^2 \to \mathbb{R}^2)$

$D\varphi = \begin{pmatrix} 1 & 0 \\ \partial_x f & \partial_y f \end{pmatrix}$ $\quad \det(D\varphi) = \partial_y f$

$\det(D\varphi_{(a,b)}) = \partial_y f(a,b) \neq 0$

$\Rightarrow D\varphi_{(a,b)}$ is inv.

$\Rightarrow$ Inv fn thm: $\exists$ small open sets on which the φ is inv!



$\Rightarrow$ on this small open set x & y can be expressed as C^1 fn's of u & v.

$$\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{\varphi} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x \\ f(x,y) \end{pmatrix}$$

① $x = u$

② y = some fn of u & v
$= h(u,v)$ ($\exists$ a C^1 fn h).

On the set $\{f(x,y) = c\}$ $\rightarrow$ Know $v = c$

$$\Rightarrow y = h(u,c) = \underbrace{h(x,c)}_{\text{what we wanted!!}}$$

$$\boxed{\text{Chose } g(x) = h(x,c)}$$

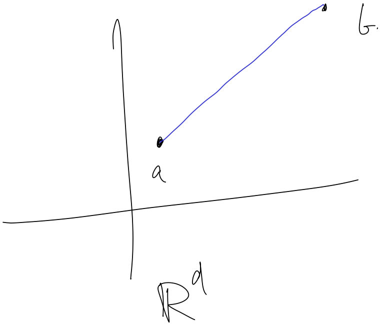
Parametrize something: (1) lines are easy

Param the line joining a & b.
(starting from a)

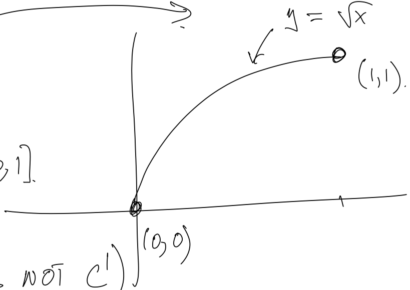
$$\gamma(t) = (1-t)a + tb$$

$\gamma(0) = a \quad \gamma(1) = b$

γ is linear.



Parametrize $y = \sqrt{x}$



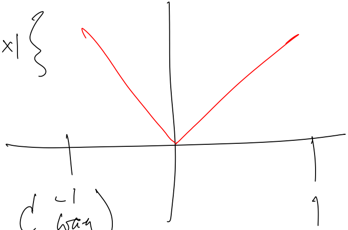
$$\gamma(t) = \begin{pmatrix} t \\ \sqrt{t} \end{pmatrix} \quad t \in [0,1]$$

(Not a C^1 param! γ is NOT C^1)

γ is not diff at 0

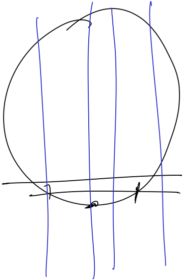
Instead: $x = y^2$. Try: $\gamma(t) = \begin{pmatrix} t^2 \\ t \end{pmatrix} \quad t \in [0,1]$

Q: $\Gamma = \{ y = |x| \}$



param Γ.

Impossible (in a C^1 way).



$$\nabla f(a) = \sum \underbrace{\lambda_i} \nabla g_i(a)$$

called Lagrange multipliers

Eg: $x, y \in \mathbb{R}^d$. $x \cdot y \leq |x|\,|y|$ (Cauchy-Schwartz)

Fix $y \in \mathbb{R}^d$.

Let $f(x) = x \cdot y$

Max f under the constr $|x| = 1$.

Solve via lag: Constr $g(x) = 1$, where $g(x) = |x|^2$

At local max/min. Know

$\exists \lambda$ ∋ $\nabla f(a) = \lambda \nabla g(a)$ $\left(\& \; g(a) = 1 \Leftrightarrow |a| = 1 \right)$

Solve: $\nabla f = \nabla(x \cdot y) = \nabla\left(\sum_{i=1}^{d} x_i y_i\right) = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_d \end{pmatrix} = y$

diff wrt x.

$\nabla(x_1 y_1 + x_2 y_2 + \cdots x_d y_d)$

$$\nabla g = \nabla\left(\sum_{i=1}^{d} x_i^2\right) = \begin{pmatrix} 2x_1 \\ 2x_2 \\ \vdots \\ 2x_d \end{pmatrix} = 2x.$$

At a: $\exists \lambda \in \mathbb{R} \ni \nabla f(a) = \lambda \nabla g(a)$

$$\Leftrightarrow y = \lambda(2a) \Leftrightarrow y = (2\lambda) a.$$

Constr: $|a|^2 = 1 \Rightarrow |a|^2 = \frac{|y|^2}{4\lambda^2} \Leftrightarrow \lambda = \pm \frac{|y|}{2}$

At a : $\lambda = \pm \frac{|y|}{2}$ & $a = \frac{y}{2\lambda} = \pm \frac{y}{|y|}$.

The const. max/min of the fn $f(x) = x \cdot y$ given the const $|x|=1$ is attained at $a = \pm \frac{y}{|y|}$

$$f\left(+\frac{y}{|y|}\right) = \frac{y}{|y|} \cdot y = +|y|. \longleftarrow \text{constr max}$$

$$f\left(-\frac{y}{|y|}\right) = \frac{-y}{|y|} \cdot y = -|y| \longleftarrow \text{constr min.}$$

Hence If $|x| = 1$, then $\underbrace{f(x)}_{x \cdot y} \leq f\left(\frac{y}{|y|}\right) = |y|$

$\Rightarrow$ If $|x| = 1$, then $x \cdot y \leq |y|$

(also If $|x| = 1$, then $x \cdot y \geq$ constr min. $= -|y|$.)

$\Rightarrow \quad x \cdot y \in (-|y|, |y|) \Rightarrow |x \cdot y| \leq |y|$ (if $|x| = 1$).

If $|x| \neq 1$: $\quad |x \cdot y| = \left| |x| \underbrace{\left(\frac{x}{|x|} \cdot y \right)}_{\leq |y|} \right|$

$\leq |x|\,|y| \quad \Rightarrow$ CoS.

① $M = \{ f = c \}$ & $a \in M$: Tgt sp at a is $\ker(Df_a)$

② $x \in \mathbb{R}^d$, $y \in \mathbb{R}^n$ $f : \mathbb{R}^d \longrightarrow \mathbb{R}^n$.

Γ = Graph of f = $\{ (x,y) \in \mathbb{R}^{d+n} \mid y = f(x) \}$

$(a, f(a)) \in \Gamma$

Q: Tgt space of Γ at $(a, f(a))$: $\operatorname{span}\left\{ \begin{pmatrix} e_i \\ \partial_i f \end{pmatrix} \;\middle|\; i \in \{1, \dots d\} \right\}$?

$= \{ (x,y) \mid y = Df_a x \}$

e_i = std basis $\mathbb{R}^d$

$\partial_i f = \begin{pmatrix} \partial_i f_1 \\ \partial_i f_2 \\ \vdots \\ \partial_i f_n \end{pmatrix}$

Q: Can you write Γ as $g(x,y) = 0$?

Yes : $g(x,y) = y - f(x)$

$\Rightarrow \Gamma = \{g = 0\} \Rightarrow$ Tgt space at $(a, f(a)) = \operatorname{Ker}(Dg_{a,f(a)})$

$$Dg_{a,f(a)} = \Big(\underbrace{\boxed{-Df_a}}_{\partial_{x_1}\ \partial_{x_2}\ \cdots} \quad \underbrace{\boxed{I}}_{\partial y_1\ \partial y_2\ \cdots\ \partial y_d} \Big)$$

$\operatorname{Ker}(Dg_{(a,f(a))}) = \{(x,y) \mid y = Df_a x\}$