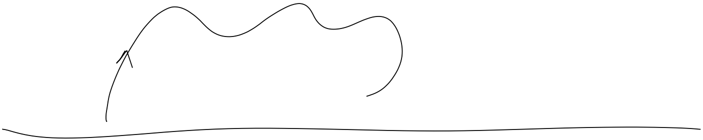
Centrifugal / centripetal force is proportional to $\frac{|v|^2}{r}$ ← Radius of curvature.



Lagrange multiplier: $f : \mathbb{R}^{n+d} \longrightarrow \mathbb{R}$, C^1

constraint.

$g : \mathbb{R}^{n+d} \longrightarrow \mathbb{R}^n$, C^1, $c \in \mathbb{R}^n$, $M = \{g = c\}$

(Assume rank $Dg_x = n$ $\forall x \in M$).

At a constr max/min a of f given the constr $\{g = c\}$

$\exists \lambda_1, \dots \lambda_n$ $\ni$

$$\nabla f(a) = \sum_{i=1}^{n} \lambda_i \nabla g_i(a)$$

Last time: (1) $\nabla f(a)$ is normal to tgt space of M at a.

(i.e. $\forall$ tgt vectors v, $\nabla f(a) \cdot v = 0$)

(2) Any normal vector is a linear comb of $\nabla g_1(a) \dots \nabla g_n(a)$

Pf of ② : $V =$ Tgt space of M at a. $= \operatorname{Ker}(Dg_a)$

$u \to$ normal vector. (i.e. $\forall v \in V$, $u \cdot v = 0$).

NTS: $\exists \lambda_1, \dots \lambda_l \ni u = \sum \lambda_i \, Dg_i(a)$.

Pf: $V = \operatorname{Ker}(Dg_a) \iff \forall v \in V,\ Dg_a v = 0$

$$\iff \begin{pmatrix} \longleftarrow \nabla g_1(a) \longrightarrow \\ \longleftarrow \nabla g_2(a) \longrightarrow \\ \vdots \\ \longleftarrow \nabla g_m(a) \longrightarrow \end{pmatrix} v = 0 \quad (\forall v \in V)$$

$\iff \nabla g_i(a) \cdot v = 0 \quad \forall i \ \&\ \forall v \in V$

$\Rightarrow$ $\nabla g_i(a)$ is a normal vector. $\forall i$

$\Rightarrow$ $\{\nabla g_1(a), \dots \nabla g_n(a)\}$ are n L.I. normal vectors.

(Rank $Dg_a = n$!)

Let $N = \{u \mid u \cdot v = 0 \;\; \forall v \in V\}$ ($N \rightarrow$ orthogonal compliment of V)

Lin alg: Know $\dim(N) + \underbrace{\dim(V)}_{d} = \underbrace{\dim(\text{whole space})}_{n+d}$.

$\Rightarrow$ $\dim(N) = n$

$\Rightarrow$ $\{\nabla g_1(a), \dots \nabla g_n(a)\}$ is a basis of N

$\Rightarrow$ any normal vect is a linear comb of $\nabla g_1(a) \cdots \nabla g_n(a)$

QED!

Wrong Proof: Max/min f given $g_1 = c_1$ & $g_2 = c_2$, ... $g_n = c_n$.

constr

Let $H(x, \lambda_1, \lambda_2, \cdots \lambda_n) = f(x) - \sum \lambda_i (g_i(x) - c_i)$

max/min H over all x & λ. (unconstrained).

at max/min H: $DH = 0$

Derivative wrt x: $Df(x) - \sum_{i=1}^{n} \lambda_i Dg_i(x) = 0$

constr is satisfied!

Diff wrt λ_i: $\frac{\partial H}{\partial \lambda_i} = 0 \iff g_i(x) - c_i = 0 \quad \forall i$

(Error: max/min of H are constr max/min of f given $g = c$.
However constr max/min of f given g NEED NOT be max/min of H).

2nd derivative test: "Bordered Hessians" gives you a test for whether the point from Lagrange mult is a max or a min.
(Not useful in practice).

$f : \mathbb{R}^{d+n} \longrightarrow \mathbb{R}$ Lag Mult: Solve $\nabla f(x) = \sum \lambda_i \nabla g_i(x)$.

$g : \mathbb{R}^{d+n} \longrightarrow \mathbb{R}^n$

$d+n$ equations in $\underbrace{d+n}_{x} + \underbrace{n}_{\text{each } \lambda}$ unknowns

$$\boxed{\nabla f(x) = \sum \lambda_i \nabla g_i(x)}$$

Constraint : $g(x) = C$ (n equations)

$d+2n$ equations in $d+2n$ unknowns.

& solve.

Typically the soln will be finitely many points

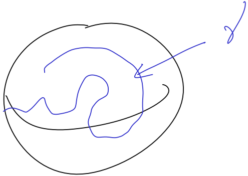
at each of these points use some ad hoc method to decide whether they are const max/min or neither.

Off γ again: $|\gamma(t)| = 1$. $v = \dot{\gamma}$ & $a = \dot{v} = \ddot{\gamma}$

(Say γ is parametrized so that $|v(t)| = 1$

NTS: $|a| \geq 1$.

(If $|v| = 1$ then curvature $= |a|$).



$$\kappa = \left(\frac{|v \times a|}{|v|^3} \right)$$

$$\kappa = \left| \frac{1}{|v|} \cdot \left(\frac{v}{|v|} \right)' \right| = |a|$$

$$|\gamma| = 1 \xRightarrow{\text{diff}} \gamma \cdot v = 0$$

$$\xRightarrow{\text{diff}} v \cdot v + \gamma \cdot a = 0$$

$$\Rightarrow |v|^2 + \gamma \cdot a = 0 \Rightarrow \gamma \cdot a = -\underbrace{|v|^2}_{1}$$

$$\Rightarrow |\gamma \cdot a| \geq 1.$$

$$\underbrace{|\gamma|}_{1} |a| \underset{\text{Cauchy Sch.}}{\geq} |\gamma \cdot a| = 1 \Rightarrow \boxed{|a| \geq 1.}$$