(1) Last time: $f : \mathbb{R}^{d+n} \longrightarrow \mathbb{R}$, C^1 fn.

$g : \mathbb{R}^{d+n} \longrightarrow \mathbb{R}^n$, $c \in \mathbb{R}^n$.

Let $M = \{g = c\}$. Goal: maximize/min f on M.

Assume: $\operatorname{rank} Dg_x = n \quad \forall x \in M$

Max/min the fn f given the constraint $g = c$

Ans: At a const local max/min, a, $\exists \lambda_1, \cdots \lambda_n \ni$

$$\nabla f(a) = \sum_{i=1}^{n} \lambda_i \nabla g_i(a)$$

Eg 1: $g(x, y) = x^2 + y^2$ & $f(x, y) = y$.

max (or min f) given constraint $g = 1$

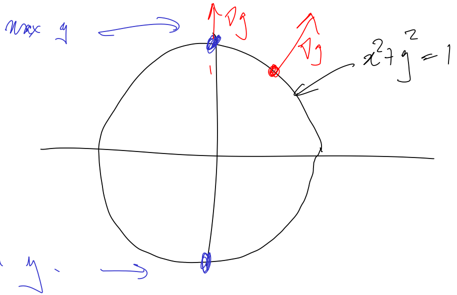
Note: $\nabla f = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow$ never 0!!

Lagrange: at max/min need

$$\nabla f(a) = \lambda \nabla g(a)$$

∇g is vertical at top & bot

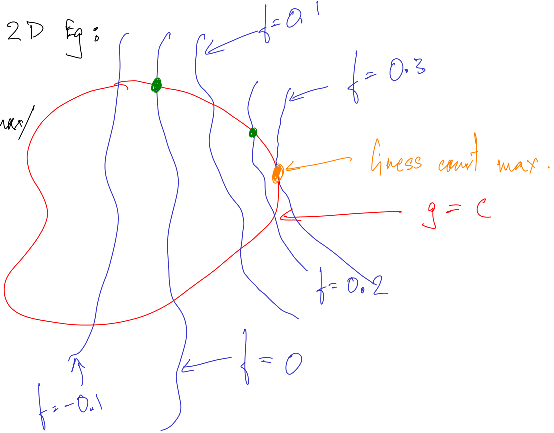
$//$ to ∇f.



Scratch for proof: 2D Eg:

Guess: At a const local max/min

we must have
The level sets of f
& the constraint $\{g = c\}$
be TANGENT.



Thm (Lagrange multipliers): $f: \mathbb{R}^{d+n} \to \mathbb{R}$, C^1, $g: \mathbb{R}^{d+n} \to \mathbb{R}^n$, C^1. $c \in \mathbb{R}^n$. $M = \{g = c\}$. Assume: $\forall x \in M$, $\operatorname{rank}(Dg_x) = n$.

Then: at a constr max/min, a, of f given the const $\{g = c\}$.

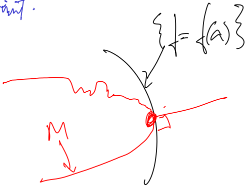
$$\exists \lambda_1, \dots \lambda_n \ \text{+} \ \nabla f(a) = \sum_{i=1}^{n} \lambda_i \nabla g_i(a)$$

Proof: Lemma 1: Say f attains a constr max/min at a.

$\underbrace{\{f = f(a)\}}$ is tgt to $\{g = c\}$ ← constraint.

Tgt space at a: $\operatorname{Ker}(Df_a)$.

∇f_a ← normal vector.



Lemma 1: At a, $\nabla f(a)$ is normal to the Tgt space of M at a.

Lemma 2: Any normal vector to tgt spce of M at a is a linear combination of $\nabla g_1(a), \cdots \nabla g_n(a)$.

Claim: we already know this

Pf of thm: at constr max/min: Lemma 1 $\Rightarrow$ $\nabla f(a)$ is normal to Tgt M at a.
Lemma 2 $\Rightarrow$ $\exists \lambda_1, \dots \lambda_n$ + $\nabla f(a) = \sum_{i=1}^{n} \lambda_i \nabla g_i(a)$.

Proof of Lemma 1: (1) Know rank $(Dg_a) = n$ by assumption.
re-order coordinates & assume the last n columns of Dg_a are L.I.

$(x, y) \in \mathbb{R}^{d+n}$, where $x \in \mathbb{R}^d$, $y \in \mathbb{R}^n$

Impl fn thm $\Rightarrow \exists$ a C^1 fn h such that $g(x,y) = c$
$\Leftrightarrow y = h(x)$ (close to the pt a).

If f has a constr max/min at a, then the fn $f(\underbrace{x, h(x)}_{\in M})$ has an unconstrained local max/min at $x = a'$.

$$\Rightarrow D\big(f(x, h(x))\big)_a = 0 \qquad \Big(a = (\underbrace{a'}_{d\text{- coords}}, \underbrace{b'}_{n\text{ - coords}})\Big)$$

$$\Rightarrow Df_a \begin{pmatrix} \boxed{I} \\ \boxed{Dh_{a'}} \end{pmatrix} = 0$$

$d \times d$ block.

$n \times d$ block.

Let $e_1, \dots e_d$ be the std basis of $\mathbb{R}^d$.

$\left\{ \begin{pmatrix} e_i \\ \partial_i h \end{pmatrix} \;\middle|\; i = 1 \dots d \right\}$ basis of the Tgt space of the graph of h.

$= \text{constraint } g.$

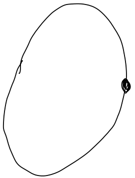
$$\Rightarrow \quad \nabla f(a)^T \begin{pmatrix} e_i \\ \partial_i h \end{pmatrix} = 0 \quad \Rightarrow \quad \nabla f(a) \cdot \begin{pmatrix} e_i \\ \partial_i h(a') \end{pmatrix} = 0$$

$\Rightarrow$ $\nabla f(a)$ is normal to the tgt space of M at a.

QED (Lemma 1)

← n colums.

$$Dg_a = \left.\underbrace{\begin{pmatrix} \downarrow & \downarrow & \downarrow & \downarrow \\ & & & \\ & & & \end{pmatrix}}_{d+n}\right\} n$$

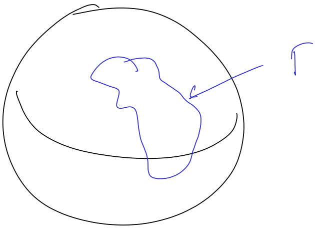


$\gamma : (0,1) \to S^2.$

$|\gamma(t)| = 1.$

$\hookrightarrow$ Curvature of γ :

$\frac{d}{dt} |\gamma(t)| = 0$



$\frac{|\gamma'' \times \gamma'|}{|\gamma'|^2} = \frac{a \times v}{|v|^2}$ Want $\geq 1.$

$$0 = \partial_t |\gamma(t)| = \partial_t \sqrt{\sum \gamma_i(t)^2} = \frac{1}{2\sqrt{\sum \gamma_i(t)^2}} \sum 2\gamma_i(t)\gamma_i'(t)$$

$$= \frac{\gamma(t) \cdot \gamma'(t)}{\underbrace{|\gamma(t)|}_{1.}} \Rightarrow \gamma \cdot \gamma' = 0$$

$$\Rightarrow \gamma(t) \cdot \gamma'(t) = 0 \ \forall t$$

diff again !!