21-268 Multidimensional Calculus: Final.

2020-05-07

- *This is an open book test. You may use your notes, homework solutions, books, and/or online resources (including software) while doing this exam. You may* not, *however, seek or receive assistance from a live human during the exam. This includes in person assistance, instant messaging, and/or posting on online forums / discussion boards. (Searching discussion boards is OK, though.)*
- *The exam has a total of 8 questions and 40 points. You have 3 hours to complete the final. When you are finished, please scan and turn in your exam via Gradescope.*
- *Your time starts as soon as you click the link to download the exam. You have 20 minutes grace time to allow for technical issues related to scanning / download. After that you will be penalized at the rate of 1% per minute for the first 5 minutes, and 10% per minute after that.*
- *If a question you are asked, or a result you want to use, has been done in class / the homework, you do* not *have to re-derive it. Simply reference it appropriately, quote/use it as needed.*
- *The questions are roughly in the order the material was covered. Difficulty wise, roughly $Q2 \leqslant Q1 \approx Q5 \leqslant Q6 \approx Q8 \leqslant Q4 < Q7 < Q3$. However this may vary based on your intuition.*
- *Good luck, and have a nice summer.*

$\boxed{5}$ 1. Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^4}$ exist? Prove it.

$\boxed{5}$ 2. Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a C^1 function such that

$$\nabla f(x) = \frac{1}{|x|^2 - 1} \begin{pmatrix} 2x_1 \\ 1 \end{pmatrix}$$

Define $g(s,t) = s^2 f(3s + t^2, s^3 - 2t)$. Find $\partial_t g(2,3)$.

$\boxed{5}$ 3. Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ be differentiable. Given $h \in \mathbb{R}$ and $a = (a_1, a_2) \in \mathbb{R}^2$, let $\Delta_h \subseteq \mathbb{R}^3$ be the triangle with vertices $\varphi(a_1, a_2)$, $\varphi(a_1 + h, a_2)$, $\varphi(a_1, a_2 + h)$. Show that

$$\lim_{h\to 0} \frac{1}{h^2} \operatorname{area}(\Delta_h)$$

exists, and express its value in terms of a, $\varphi(a)$ and derivatives of φ at a. (Partial credit for correctly computing the limit without a completely rigorous derivation.)

4. Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a C^2 function, $c \in \mathbb{R}$, and $S \subseteq \mathbb{R}^3$ be the surface $\{f = c\}$. Suppose further for all $x \in S$, $\operatorname{rank} Df_x = 1$.

$\boxed{2}$ (a) Show that there exists a *continuous* function $\hat{n}\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that for every $x \in S$, $\hat{n}(x)$ is a unit normal vector to the surface S at the point x.

$\boxed{3}$ (b) For a fixed $a \in S$, let $u = \nabla f(a)$ and $A = Hf_a$. Find $|\nabla \cdot \hat{n}(a)|$ in terms of u and A.
You may use things like Au, $\det A$ or $\operatorname{trace}(A)$ in your answer. You may also write $A = (a_{i,j})$, and write down a formula using sums involving u_i and $a_{i,j}$. You may not, however, have things like $\partial_i u_j$ in your answer. You should simplify so that your answer is expressed in terms of u and A, without involving further derivatives.

$\boxed{5}$ 5. Maximize $x^2 + y^2 + z^2$ subject to the constraints $x + 2y + 3z = 1$ and $z = x^2 + y^2$.

$\boxed{5}$ 6. Evaluate $\displaystyle\int_{z=0}^{1} \int_{y=0}^{1} \int_{x=y^2}^{1} \sqrt{xz}\, e^{-x} \, dx \, dy \, dz$.

$\boxed{5}$ 7. Let $U = \{x \in \mathbb{R}^2 \mid x \neq 0\}$, and define $f\colon U \to \mathbb{R}$ by $f(x) = \ln|x|$. Let $V \subseteq \mathbb{R}^2$ be any bounded open set such that $0 \in V$ and the boundary of V is a finite union of piecewise C^1 curves. For each point on ∂V, let $\hat{n}$ denote a unit normal vector that points outwards to V. Compute

$$\oint_{\partial V} \nabla f \cdot \hat{n} \, |d\ell| \,.$$

$\boxed{5}$ 8. Let $R > 0$, and $\Sigma \subseteq \mathbb{R}^3$ be the hemisphere consisting of all points $x \in \mathbb{R}^3$ such that $|x| = R$ and $x_3 > 0$. Orient Σ by choosing $\hat{n}$ to be the *upward pointing* unit normal (i.e. for every point $x \in \Sigma$, we choose $\hat{n}(x)$ be the unit normal vector for which $\hat{n}(x) \cdot e_3 > 0$, where $e_3 = (0, 0, 1)$). Let $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $F(x) = (x_1 + x_2, x_1, 1 + x_3)$. Compute $\int_\Sigma F \cdot \hat{n} \, dS$.