Joint QV:

$$X(t) = \int_0^t \sigma(s)\, dM(s) \quad , \quad Y(t) = \int_0^t \tau(s)\, dN(s)$$

$$\Rightarrow [X, Y](t) = \int_0^t \sigma(s)\, \tau(s)\, d[M, N](s).$$

Q: Can you find $\tilde{P}$ such that $t\,W(t)$ is a BM under $\tilde{P}$.

Guess $\boxed{\text{NO}}$

① Obs: Say P, $\tilde{P}$ 2 equiv measures.

$X \longrightarrow$ some Ito process.

$$[X,X](T) = \lim_{\|P\| \to 0} \sum \left(X(t_{i+1}) - X_{t_i} \right)^2$$

$\hookrightarrow$ $[X,X](T)$ is independent of the prob measure.

$\hookrightarrow \Rightarrow$ $[X,X](T)$ under $\tilde{P}$ $=$ $[X,X](T)$ under P.

Let $X(t) = tW(t)$.

$\cancel{[X,X]:}$ $dX = t\,dW + W\,dt$

$\Rightarrow d[X,X] = t^2\,dt \Rightarrow [X,X](t) = \frac{t^3}{3} \neq t$.

$\Rightarrow$ there does not exist a equiv meas $\tilde{P}$ under which X is a B.M.

$\boxed{4.5}$ (a) $X \to$ Ito process.

$$dX = \theta(\mu - X(t))\, dt + \sigma\, dW \;; \quad X(0) = x_0.$$

(a) find f & g $\ni$

$$X(t) = f(t) + \int_0^t g(s,t)\, dW(s).$$

~~$X = e^{-\theta t} X(t)$~~ Trick: Compute $d\left(e^{+\theta t} X(t)\right)$.

$$d(e^{\theta t} X) = e^{\theta t} dX + \theta X e^{\theta t} dt$$

$$= e^{\theta t}\left(\theta(\mu - \cancel{X}) dt\right) + e^{\theta t} \sigma\, dW + \cancel{\theta X e^{\theta t} dt}$$

$$= \mu\theta e^{\theta t} dt + e^{\theta t} \sigma\, dW.$$

$$\Rightarrow e^{\theta t} X(t) - x_0 = \mu\left(e^{\theta t} - 1\right) + \int_0^t e^{\theta s} \sigma \, dW(s).$$

$$\Rightarrow X(t) = \underbrace{e^{-\theta t} x_0 + \mu\left(1 - e^{-\theta t}\right)}_{f} + \int_0^t \underbrace{e^{-\theta(t-s)} \sigma}_{g} \, dW(s).$$

(b) Compute $E X(t)$, $\operatorname{cov}(X(s), X(t))$.

① $E X(t) = e^{-\theta t} x_0 + \mu\left(1 - e^{-\theta t}\right).$

② $\cancel{\operatorname{cov}(X(t) X(s), X(t))}$.

$$E\int_0^t e^{-\theta(t-s)}\sigma\, dW(s) = ?$$

① $M(t) = \int_0^t \underbrace{e^{-\theta(t-s)}\sigma}\, dW(s)$ need not be a mg!

② But $EM(t) = E\int_0^t e^{-\theta(t-s)}\sigma\, dW(s) =$

$$= e^{-\theta t}\, \underbrace{E\int_0^t e^{\theta s}\sigma\, dW(s)}_{\text{mg}} = 0$$

If $N(t) = \int_0^t f(s)\, dW(s) \Rightarrow N$ is a mg,

($f(s)$ adapted)

$M(t) = \int_0^t g(s,t)\, dW(s) \longleftarrow$ Need not be a mg.

Compute $\operatorname{cov}(X(s), X(t))$.

$$= E\big(X(s) - EX(s)\big)\big(X(t) - EX(t)\big).$$

$$= E\left(\cancel{e^{-\theta s}} \int_0^s e^{-\theta(s-r)}\sigma\, dW(r)\right)\left(\int_0^t e^{-\theta(t-r)}\sigma\, dW(r)\right).$$

$$= \sigma^2 e^{-\theta(s+t)} E\left(\int_0^s e^{\theta r}\, dW(r) \int_0^t e^{\theta r}\, dW(r)\right).$$

$$= \sigma^2 e^{-\theta(s+t)} E\left(\left(\int_0^s e^{\theta r}\, dW(r)\right)^2 + \left(\int_0^s e^{\theta r}\, dW(r)\right)\left(\int_s^t e^{\theta r}\, dW(r)\right)\right).$$

$$\boxed{= \sigma^2 e^{-\theta(s+t)} \int_0^s e^{2\theta r}\, dr} + 0$$

$\boxed{4.10}$ $M \to$ cts process. M & M^2 are both Mg's.

Q: Must M be a constant?

Guess Yes:

Set $N(t) = M(t) - M(0)$.

N & N^2 are mg's.

Know $N^2 - [N,N]$ is a mg
Given $N^2 - 0$ is a mg
} Uniqueness $\Rightarrow [N,N] = 0$

$$E N(t)^2 - \underbrace{E N(0)^2}_{0} = E[N,N](t) = 0 \Rightarrow E N(t)^2 = 0$$

$$\Rightarrow N(t) = 0 \text{ a.s.}$$

$\boxed{7.4}$ $M(t) = \int_0^t W(s)\, dW(s)$.

Find a fn f such that $E(t) = \exp\left(M(t) - \int_0^t f(s, W(s))\, ds\right)$

is a mg.

General rule: $X(t) = \exp\left(M(t) - \frac{1}{2}[M,M](t)\right)$ is a mg.

Check: Compute dX:

$$dX = -\frac{1}{2}\cancel{X(t)\, d[M,M](t)} + X(t)\, dM(t)$$
$$+ \frac{1}{2} X \cancel{d[M,M](t)}$$

Let $f(t,x) = e^{x - \frac{1}{2}[M,M](t)}$.

$X = f(t, M)$

$\Rightarrow dX = \partial_t f \; dt + \partial_x f \; dM + \frac{1}{2} \partial_x^2 f \; d[M,M]$.

$$= \underbrace{e^{M(t) - \frac{1}{2}[M,M](t)}}_{X} \left(-\frac{1}{2}\right) d[M,M] + X\,dM + \frac{1}{2} X\, d[M,M]$$

$$= X\,dM.$$

$\Rightarrow X$ is a mg.

Back to problem: Want $\int_0^t f(s, W(s))\,ds = \frac{1}{2} \int_0^t W(s)^2\,ds$.

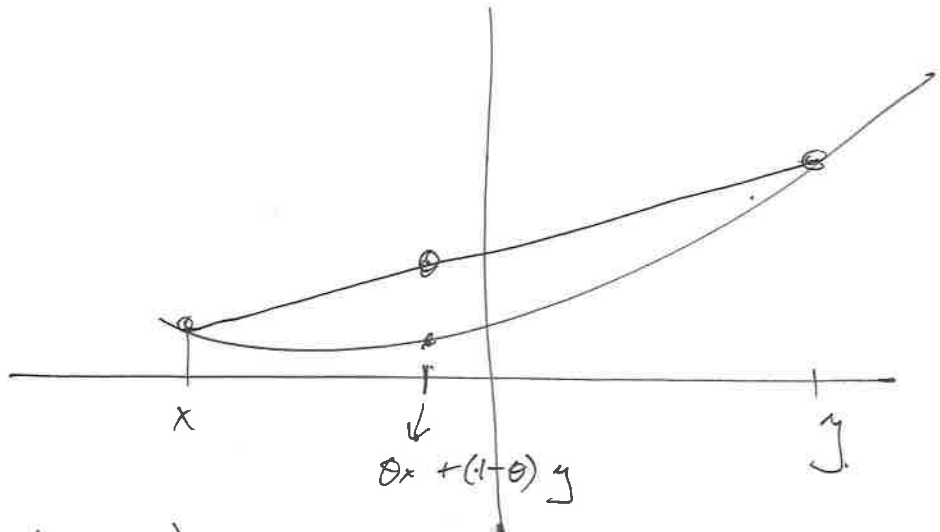
Choose $f(s) = W(s)^2/2$,

$\boxed{4.6}$ M is a Mg (cts). φ is a convex function $\Big\}$ Q: Is $\varphi(M)$ a sub mg?

YES: ① Jensen.

② Itô:

Convex $\Rightarrow$ φ' is inc
$\Rightarrow \varphi'' \geq 0$



Sub Mg: ~~$E(M(t) \mid \mathcal{F}_s) \geq M(s)$. ← Need to~~ to.

Need to show $E(\varphi(M(t)) \mid \mathcal{F}_s) \geq \varphi(M(s))$.

$$d(\varphi(M(t))) = \varphi'(M) \, \cancel{dt}\, dM + \frac{1}{2} \varphi'' \, d[M, M].$$

$$\Rightarrow \varphi(M(t)) = \varphi(M(s)) + \int_s^t \varphi'(M(r))\, \cancel{dr}\, dM(r) + \frac{1}{2}\int_s^t \underbrace{\varphi''(r)}_{\geq 0}\, \underbrace{d[M,M](r)}_{\geq 0}.$$

$$\Rightarrow E\left(\varphi(M(t)) \,\middle|\, \mathcal{F}_s\right) = \varphi(M(s)) + 0 + \boxed{+ve}$$

$$\Rightarrow E\left(\varphi(M(t)) \,\middle|\, \mathcal{F}_s\right) \geq \varphi(M(s)) \Rightarrow \varphi(M) \text{ is a sub mg}.$$

Jensen Ineq: (HW 1).

$\varphi \rightarrow$ convex fn. X R.V.

$E\,\varphi(X) \geq \varphi(EX)$: Convex $\Leftrightarrow \varphi(\theta x + (1-\theta)y) \leq \theta\varphi(x) + (1-\theta)\varphi(y)$.

Claim: φ convex then $E(\varphi(X) \mid \mathcal{F}) \geq \varphi(E(X \mid \mathcal{F}))$ (Conditional Jensen)

M is a mg & φ is convex.

$$E\big(\varphi(M(t))\,\big|\,\mathcal{F}_s\big) \geq \varphi\Big(\underbrace{E\big(M(t)\,|\,\mathcal{F}_s\big)}_{M(s)}\Big).$$

$$= \varphi(M(s)). \Rightarrow \varphi(M(t)) \text{ is a sub mg.}$$

4.7) $Z(t) = \exp\left(\theta W(t) - \frac{\theta^2}{2} t\right)$.

Given $0 \leq s \leq t$, and a fn f.

Find g such that $E\big(f(Z(t))\,\big|\,\mathcal{F}_s\big) = g(Z(s))$.

$$E\left(f\left(Z(t)\right) \mid \mathcal{F}_s\right) = E\left(f\left(Z(s)\exp\left(\theta\left(W(t)-W(s)\right) - \frac{\theta^2}{2}(t-s)\right)\right) \mid \mathcal{F}_s\right)$$

$$= \int \frac{e^{-y^2/t-s}}{\sqrt{2\pi(t-s)}} f\left(Z(s)\exp\left(\theta y - \frac{\theta^2}{2}(t-s)\right)\right) dy$$

2017 Final Q8

Compute $\lim_{t \to s^+} \frac{1}{t-s}\left(E\left(W(t)^{\alpha} \mid \mathcal{F}_s\right) - W(s)^{\alpha}\right)$, $\alpha \geq 2$.

① ~~Fx~~ Compute $E\left(W(t)^{\alpha} \mid \mathcal{F}_s\right)$:

Ito: $d\left(W(t)^{\alpha}\right) = \alpha W(t)^{\alpha-1} dW + \frac{1}{2}\alpha(\alpha-1) W(t)^{\alpha-2} dt$

$\Rightarrow W(t)^{\alpha} = W(s)^{\alpha} + \int_s^t \alpha W(r)^{\alpha-1} dW(r) + \frac{\alpha(\alpha-1)}{2} \int_s^t W(r)^{\alpha-2} dr.$

$\Rightarrow \frac{1}{t-s}\left(E\left(W(t)^{\alpha} \mid \mathcal{F}_s\right) - W(s)^{\alpha}\right) =$

$$= \frac{\alpha(\alpha-1)}{2(t-s)} \int_s^t E\left(W(r)^{\alpha-2} \,\middle|\, \mathcal{F}_s\right) dr .$$

$$\Rightarrow \lim_{\substack{\cancel{s \to t^+} \\ t \to s^+}} \frac{1}{t-s} \left(\quad \right) = \lim_{\substack{\cancel{s \to t} \\ t \to s^+}} \frac{\alpha(\alpha-1)}{2(t-s)} \int_s^t E\left(W(r)^{\alpha-2} \,\middle|\, \mathcal{F}_s\right) dr .$$

$$= \frac{\alpha(\alpha-1)}{2\cancel{(t-s)}} E\left(W(s)^{\alpha-2} \,\middle|\, \mathcal{F}_s\right) .$$

$$= \frac{\alpha(\alpha-1)}{2} W(s)^{\alpha-2} .$$

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2017 Final Q6

Compute $E\left|\int_0^t W(s)\,ds\right|^{1/2}$.

Let $X(t) = \int_0^t W(s)\,ds$. Know X is normal.

$EX(t) = 0$.

$EX(t)^2 = E\left(\int_0^t W(s)\,ds \cdot \int_0^t W(r)\,dr\right) = \underbrace{\int_0^t \int_0^t (s \wedge r)\,ds\,dr}_{\text{simplify}}$.

Let $\sigma^2 = EX(t)^2 = \left(\frac{t^3}{3}\right)$.

$$E\left|\int_0^t W(s)\,ds\right|^{1/2} = E\,|N(0,\sigma^2)|^{1/2}.$$

$$= \int_{y\in\mathbb{R}} \frac{e^{-y^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,|y|^{1/2}\,dy.$$

$\boxed{Q5}$ Stock $\longrightarrow$ GBM(α, σ).

M.M $\longrightarrow r$

$K_1, K_2, T > 0$.

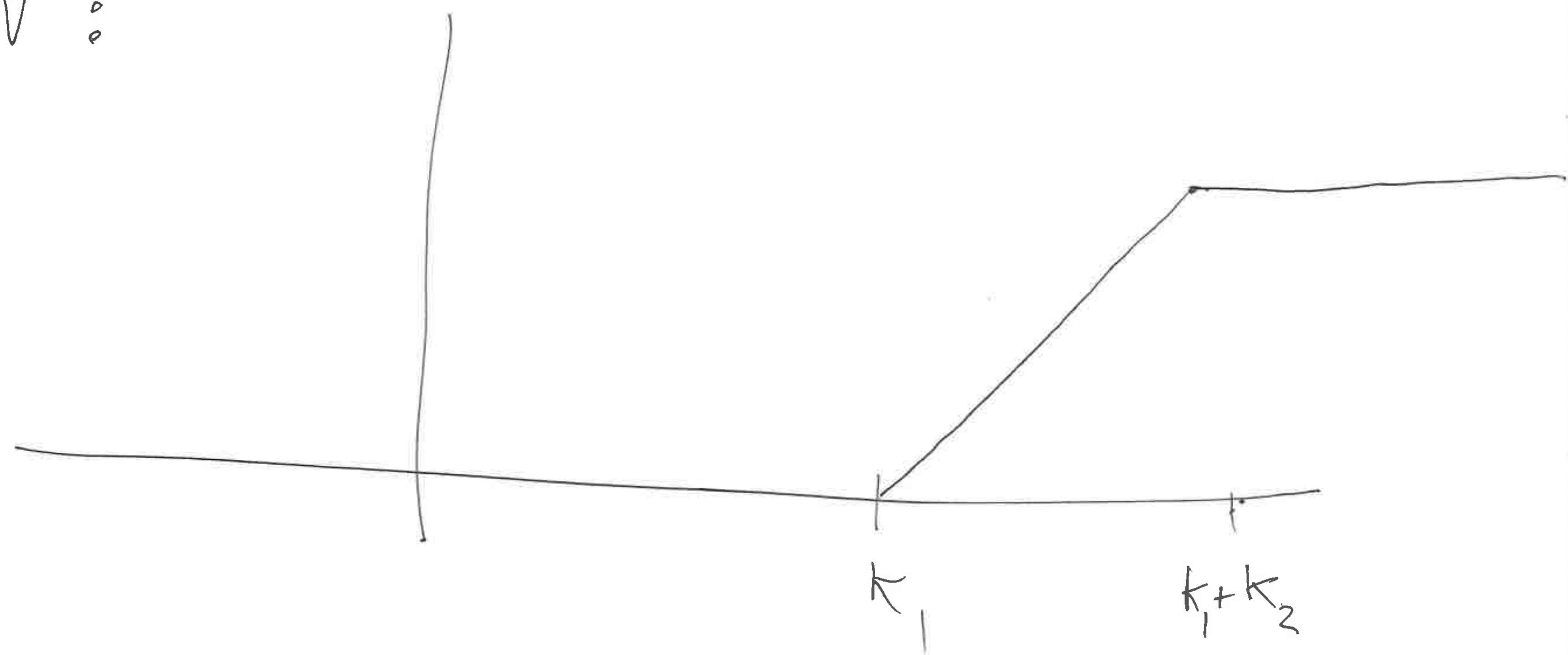
$$V(T) = \min\{(S(T)-K_1)^+, K_2\}.$$

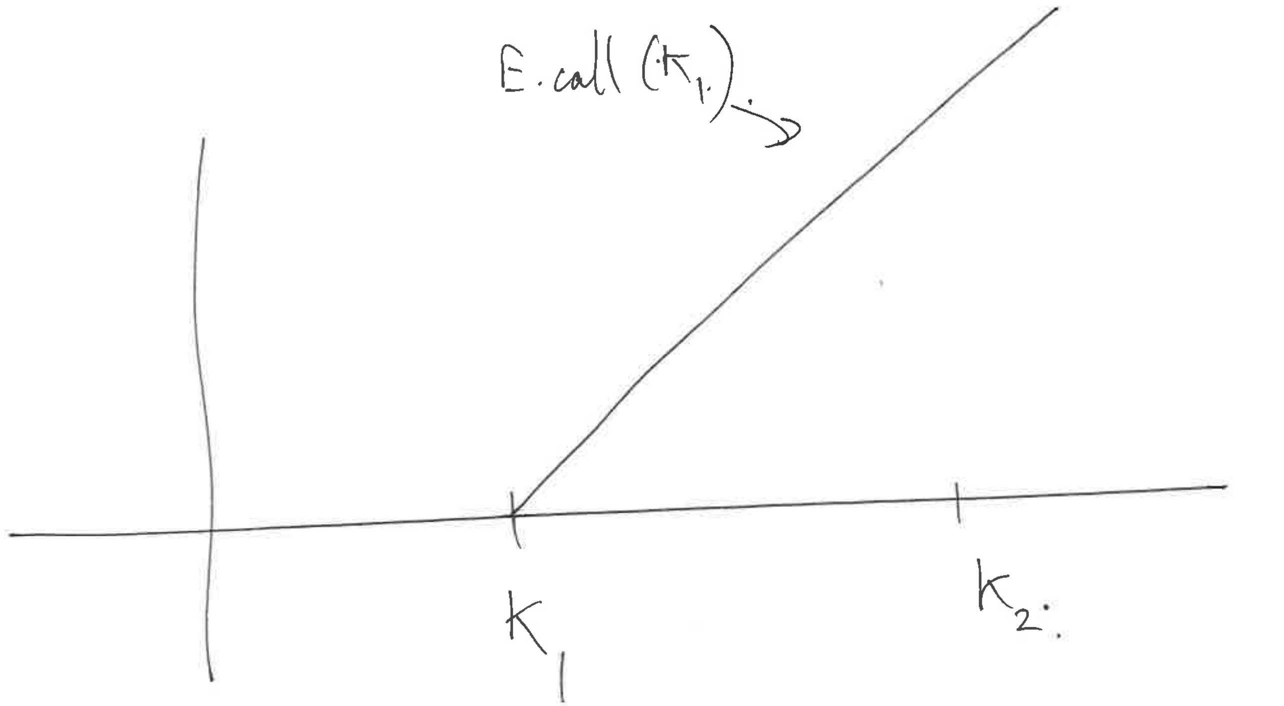
Security pays $V(T)$ at time T. Price it.

Sol: (1) RNP formula:

$$V(t) = \tilde{E}\left(e^{-r(T-t)}\left((S(T)-K_1)^+ \wedge K_2\right) \Big| \mathcal{F}_t\right).$$

(2) Draw V:





$$(x-K_1)^+ \wedge K_2 = (x-K_1)^+ - (x-(K_1+K_2))^+$$

$W_1, W_2 \to 2$ BM's.

$\cancel{dW}$ $d[W_1, W_2] = \rho \, dt$. $\quad (\rho \in (-1, 1))$.

$\theta_1, \theta_2 \to 2$ constants.

$\tilde{W}_1 = \theta_1 t + W_1(t)$, $\quad \tilde{W}_2 = \theta_2 t + W_2(t)$.

Find $Z > 0$, $E Z(T) = 1$. $\quad d\tilde{P} = Z(T) \, dP$.

Want $\tilde{W}_1$ & $\tilde{W}_2$ to be B.M's under $\tilde{P}$.

& $\boxed{d[\tilde{W}_1, \tilde{W}_2] = \rho}$ ← free.

Guess work:

?

$$dZ = -Z \left(\quad \right).$$

Want $\tilde{W}_1$ to be a mg under $\tilde{P}$.

$\iff$ $\tilde{W}_1 Z$ is a mg under P.

Compute $d(\tilde{W}_1 Z) = \tilde{W}_1 dZ + Z d\tilde{W}_1 + d[\tilde{W}_1, Z]$.

$$= \tilde{W}_1 dZ + Z dW_1 \underbrace{+ Z\theta_1 dt + d[\tilde{W}_1, Z]}.$$

Want $\boxed{Z\theta_1 dt + d[\tilde{W}_1, Z] = 0}$

Same for $\tilde{W}_2$: Want $\boxed{Z\theta_2 dt + d[\tilde{W}_2, Z] = 0}$

Want $d[z, \tilde{W}_1] = -z\theta_1 \, dt$
& $d[z, \tilde{W}_2] = -z\theta_2 \, dt$

Say $dz = -z\left(\alpha_1 \, d\tilde{W}_1 + \alpha_2 \, d\tilde{W}_2\right)$.

Then $d[z, \tilde{W}_1] = -z(\alpha_1 + \alpha_2 \rho) \, dt$
& $d[z, \tilde{W}_2] = -z(\alpha_2 + \alpha_1 \rho) \, dt$.

Need $\left.\begin{aligned} \alpha_1 + \rho\alpha_2 &= \theta_1 \\ \alpha_2 + \rho\alpha_1 &= \theta_2. \end{aligned}\right\}$ Solve for α_1 & α_2.

$\Rightarrow z(t) = \exp\left(-\alpha_1 \tilde{W}_1 - \alpha_2 \tilde{W}_2 - \frac{1}{2}\left(\alpha_1^2 + \alpha_2^2 + 2\rho\alpha_1\alpha_2\right) t\right)$ //