→ Final → Mar 6th (5:30 PM)

NY: Steve Shreve → Office Hours 2:30 PM (Sat).

Pitt: Me Office Hours 2:30 PM (Friday).

<u>Next week:</u>. Hope 2:45 PM (~~Pitt~~)(Tue) → Problem session (Here).
(Maybe).

<u>FCE:</u>. Incentive: 80% Response Rate by TOMORROW

⟹ Final Grades Released ASAP.

(If not Final Grades on March 13th 5:00 PM)

Response rate now: 50%.

Q1: Let $X(t) = \int_0^t f(s)\, dW(s)$

$f \longrightarrow$ Non Random fn.

Find dist of X.

Sol: Know X is normal (sum of Joint Normals is Normal. & lim of Normals is ↑).

Compute $EX(t)$ & $EX(t)^2$.

① $EX(t) = 0$

② $EX(t)^2 = \int_0^t f(s)^2\, ds$. (Itô Isometry).

Variant: Q: $X(t) = \int_0^t f(s) W(s)\, ds$.

Compute the dist of X.

f not random.

Sol: $X(t) = \lim_{\|P\| \to 0} \underbrace{\sum f(t_i) W(t_i)(t_{i+1} - t_i)}_{\text{Normal}}$.

Expect $X(t) \sim N(\mu, \sigma^2)$. Compute μ & σ.

① $EX(t) = E \int_0^t f(s) W(s)\, ds = \int_0^t f(s) (EW(s))\, ds = 0$,

(2) $$\mathbb{E}\, X(t)^2 \overset{\text{trick 1}}{=} \mathbb{E}\left(\int_0^t f(s)W(s)\,ds\right)\left(\int_0^t f(\overset{r}{\cancel{s}})W(\overset{r}{\cancel{s}})\,d\overset{r}{\cancel{s}}\right)$$

$$= \mathbb{E}\int_0^t\int_0^t f(s)f(r)W(s)W(r)\,ds\,dr.$$

$$= \int_{s=0}^{t}\int_{r=0}^{t} f(s)f(r)\,(s\wedge r)\,ds\,dr.$$ & compute this.

② Trick 2: $dX = f(t)\,W(t)\,dt$.

Let $F(t) = \int_0^t f(s)\,ds$ $\quad (F' = f)$.

$$d\big(F(t)W(t)\big) = F\,dW + \underbrace{W \cdot f\,dt}_{dX} + \overbrace{d[F,W]}^{0}.$$

$$\Rightarrow F(t)W(t) = \int_0^t F(s)\,dW(s) + \underbrace{\int_0^t W(s)f(s)\,ds}_{X(t)},$$

$$\Rightarrow X(t) = F(t)W(t) - \int_0^t F(s)\,dW(s).$$

$$\Rightarrow X(t)^2 = F(t)^2 W(t)^2 + \left(\int_0^t F(s)\, dW(s) \right)^2$$

$$\cancel{+} - 2F(t) W(t) \int_0^t F(s)\, dW(s).$$

$$\Rightarrow E\, X(t)^2 = F(t)^2 t + \int_0^t F(s)^2\, ds - 2F(t)\, E\left(W(t) \int_0^t F(s)\, dW(s) \right),$$

(Itô Isom). $\quad (*)$.

Let $M(t) = \int_0^t F(s)\, dW(s)$. Compute $E\, W(t) M(t)$.

$$d(WM) = W\, dM + M\, dW + d\,[W, M].$$

$$= W\, dM + M\, dW + F(t)\, dt$$

$$\left(\because \begin{array}{l} dW = 1 \, dW \\ dM = F(t)\, dW \end{array} \right\} d[W,M]_t = 1\, F(t)\, d[W,W] \right)$$

$$\Rightarrow W(t)M(t) = \int_0^t W(s)\,dM(s) + \int_0^t M(s)\,dW(s) + \int_0^t F(s)\,ds.$$

$$\Rightarrow E\, M(t)W(t) = 0 + 0 + \int_0^t F(s)\,ds.$$

& substitute in $(*)$ for an answer!

Q: $Y(t) = \int_0^t f(s)\,W(s)\,dW(s)$. is Y Normal?

NO: $f(t_i)\,W(t_i)\,(W(t_{i+1}) - W(t_i))$, not normal. ~~can be an~~

$\underline{Q2:}$ $M, N \to 2$ mg (cts).

$$\left.\begin{aligned} d[M,M] &= \sigma(t)\,dt \\ d[N,N] &= \tau(t)\,dt \\ d[M,N] &= \rho(t)\,dt. \end{aligned}\right\}$$ σ, τ, ρ are non-random.

(a) Compute $E \exp\big(\lambda M(t) + \mu N(t)\big)$.

Sol: $\underline{\text{Ito:}}$ $Y(t) = e^{\lambda M(t) + \mu N(t)} = g(M(t), N(t))$.

where $g(x,y) = e^{\lambda x + \mu y}$.

$$\partial_x g = \lambda g, \quad \partial_y g = \mu g. \qquad \partial_x^2 g = \lambda^2 g \quad \partial_y^2 g = \mu^2 g$$

$$\partial_x \partial_y g = \lambda \mu g.$$

$$\Rightarrow dY = \partial_x g \, dM + \partial_y g \, dN + \frac{1}{2}\Big(\partial_x^2 g \, d[M,M] + \partial_y^2 g \, d[N,N] + 2\partial_x\partial_y g \, d[M,N] \Big).$$

$$\Rightarrow dY = \lambda Y \, dM + \mu Y \, dN + \frac{1}{2}\left[\lambda^2 Y \sigma + \mu^2 Y \tau + 2\lambda\mu Y \rho \right] dt.$$

$$\Rightarrow Y(t) = 1 + \int_0^t \lambda Y(s) \, dM(s) + \int_0^t \mu Y(s) \, dN(s) + \frac{1}{2}\int_0^t (\lambda^2\sigma + \mu^2\tau + 2\lambda\mu\rho) Y \, ds$$

Let $f(t) = E\, Y(t)$

$$\Rightarrow f(t) = 1 + 0 + 0 + \frac{1}{2}\int_0^t (\lambda^2\sigma + \mu^2\tau + 2\lambda\mu\rho) f(s) \, ds.$$

$$\Rightarrow \cancel{f(t)}\ \partial_t f = \frac{1}{2}\left(\lambda^2\sigma + \mu^2\tau + 2\lambda\mu\rho\right) f$$

$$\boxed{\Rightarrow f(t) = 1 \exp\left(\frac{1}{2}\int_0^t \left(\lambda^2\sigma(s) + \mu^2\tau(s) + 2\lambda\mu\rho(s)\right) ds\right).}$$

$$\boxed{\|\ \ E \exp\left(\lambda M(t) + \mu N(t)\right).}$$

MGF of a joint normal!

(b) Suppose $\sigma(t) = \tau(t) = 1$
& $\rho(t) = 0$

Then $E \exp\left(\lambda M(t) + \mu N(t)\right) = \exp\left(\frac{1}{2}\left(\lambda^2 t + \mu^2 t\right)\right).$

the MGF of ~~2D~~ $N\left(0, \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\right)$

to show (M, N) is a 2D BM,

Just compute $E\left(\exp\left(\lambda(M(t)-M(s)) + \mu(N(t)-N(s))\right) \,\middle|\, \mathcal{F}_s\right)$.

<u>Q3</u>: Financial Market
- Risky asset. Price $GBM(\alpha, \sigma)$. (S).
- Money market: Interest rate r.

Security: Payoff at maturity T is $\frac{1}{T}\int_0^T S(s)\, ds$.

Compute the AFP at time $t \leq T$.

Sol: RNP formula: $V(T) = \frac{1}{T}\int_0^T S(s)\,ds.$

$\tilde{P} \rightarrow$ Risk Neutral measure.

$\Big(d\tilde{P} = Z(T)\,dP$ ← Forget this formula.

$\& \quad Z(t) = \exp\left(\cancel{\left(\alpha - \frac{\sigma^2}{2}\right)} \left(\frac{\alpha - r}{\sigma}\right) \cancel{t}\, W(t) - \frac{1}{2}\left(\frac{\alpha - r}{\sigma}\right)^2 \cancel{W(t)}\, t \right),$

$$V(t) = \tilde{E}\left(e^{-r(T-t)} V(T) \,\middle|\, \mathcal{F}_t \right).$$

Only use, under $\tilde{P}$, $S \sim GBM(r, \sigma)$,

$$\tilde{E}\left(\frac{e^{-r(T-t)}}{T} \int_0^T S(s)\,ds \,\middle|\, \mathcal{F}_t \right) = \tilde{E}\left((\ ) \int_0^t + \int_t^T \,\middle|\, \mathcal{F}_t \right).$$

$$= \frac{e^{-r(T-t)}}{T} \int_0^t S(s)\,ds. + \frac{e^{-r(T-t)}}{T} \int_t^T \underbrace{\tilde{E}\left(S(s) \,\middle|\, \mathcal{F}_t \right)}_{\substack{\text{compute this.} \\ (s>t).}} ds.$$

$S \to$ GBM under $\tilde{P}$
(r, σ).

$$S(s) = S(0) \exp\left(\left(r - \frac{\sigma^2}{2}\right) s + \sigma \tilde{W}(s) \right),$$

$$S(s) = S(t) \exp\left(\left(r - \frac{\sigma^2}{2}\right)(s-t) + \sigma\left(\tilde{W}(s) - \tilde{W}(t)\right)\right),$$

$$\Rightarrow \tilde{E}\left(S(s) \mid \mathcal{F}_t\right) = S(t) \exp\left(\left(r - \frac{\sigma^2}{2}\right)(s-t)\right) \cdot$$

$$\cdot \tilde{E}\left(\exp\left(\sigma\left(\tilde{W}(s) - \tilde{W}(t)\right)\right) \mid \mathcal{F}_t\right)$$

$$= S(t) \exp\left(\left(r - \frac{\sigma^2}{2}\right)(s-t)\right) \cdot \tilde{E} \exp\left(\sigma\sqrt{s-t}\; \overset{\text{std normal.}}{N}\right).$$

$$= S(t) \exp\left(\left(r - \frac{\sigma^2}{2}\right)(s-t)\right) \cdot \exp\left(\frac{\sigma^2 (s-t)}{2}\right).$$

$$= S(t) \exp\left(r\,(s-t)\right).$$

$$\therefore V(t) = \frac{e^{-r(T-t)}}{T}\int_0^t S(s)\,ds + \left(\frac{e^{-r(T-t)}}{T}\int_t^T \exp\left(r(s-t)\right)ds\right)S(t).$$

$$\boxed{V(t) = \frac{e^{-r(T-t)}}{T}\int_0^t S(s)\,ds + \frac{S(t)}{rT}\left(1 - e^{-r(T-t)}\right) \quad (**)}$$

Q: What is the trading strategy to build the R. Portfolio?

$\hookrightarrow$ $$dV = \left(\Delta(t)\right)dS + r\left(V(t) - \Delta(t)S(t)\right)dt$$

From $(**)$: $$\Delta(t) = \frac{1 - e^{-r(T-t)}}{rT}$$

Q: $\cancel{X \sim N(0,1)}$. $\cancel{a \in \mathbb{R}}$.

$\cancel{X + a \sim N(}$

RNP formula:

$$V(t) = \tilde{E}\left(\exp\left(-\int_t^T R(s)\,ds\right) V(T) \,\middle|\, \mathcal{F}_t\right)$$

Q: $W \longrightarrow$ BM.

Let $dB = \operatorname{sign}(W(t))\, dW$

$$\operatorname{sign}(x) = \begin{cases} 1 & x > 0 \\ -1 & x \le 0 \end{cases}$$

Q: Is B a BM?

Yes! ① B is a cts mg. ✓

$$\left(B(t) = \int_0^t \operatorname{sign}(W(s))\, dW(s) \right).$$

② $d[B,B](t) = \operatorname{sign}(W(t))^2\, dt = dt.$ ✓

By Levy $\Rightarrow$ B is a BM.

② Compute $[B, W]$ & $E(BW)$.

$d[B,W]$: $\longrightarrow$ $\left.\begin{array}{l} dB = \operatorname{sign}(W(t))\, dW \\ dW = 1 \quad dW \end{array}\right\}$ $\boxed{d[B,W] = \operatorname{sign}(W(t))\, dt}$

~~E(BW~~ $E(B(t)W(t))$:

$$d(BW) = B\,dW + W\,dB + d[B,W].$$

$$= B\,dW + W\,dB + \text{Sign}(W(t))\,dt.$$

$$\Rightarrow B(t)W(t) = 0 + \int_0^t B(s)\,dW(s) + \int_0^t W(s)\,dB(s) + \int_0^t \text{Sign}(W(s))\,ds.$$

$$\Rightarrow E\,B(t)W(t) = 0 + 0 + 0 = 0.$$

Q: $B(t) \sim N(0,t)$.
$W(t) \sim N(0,t)$.
$E\,B(t)W(t) = 0$

Q: Are B & W independent??

NO: If B & W are ind

$\Rightarrow [B,W] = 0$. But $[B,W] = \int_0^t \text{Sign}(W(s))\,ds$,