Today:

- Multi Dimensional Ito
- Change of Measure
- Girsanov Theorem
- RNP.

Finish off with some exam tips.

~~Ex~~

$$f(t, x_1, \dots, x_n) = f(t, \underline{x}) \in C^{1,2}$$

$$X = (X^1, X^2, \dots, X^n)$$

then if X is an Ito process

$$df(t, X_t) = \partial_t f(t, X_t) + \sum_{i=1}^{n} \partial_i f(t, X_t)\, dX_i$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_{ij} f(t, X_t)\, d[X_i, X_j]_t.$$

Ex

$$\begin{cases} dX_t = \mu_1 X_t\, dt + \sigma_1 X_t\, dW_{1,t} \\ dY_t = \mu_2 Y_t\, dt + \sigma_2 Y_t\, dW_{2,t} \end{cases}$$

$$d[W_1, W_2]_t = \rho\, dt \quad , \quad -1 \leq \rho \leq 1.$$

let $Z_t = \frac{X_t}{Y_t}$ (a) Find dZ_t.

$$f(t,x,y) = \frac{x}{y}.$$

$$f_x = \frac{1}{y}, \quad f_{xx} = 0. \qquad f_{xy} = f_{yx} = \frac{-1}{y^2}.$$

$$f_y = \frac{-x}{y^2} \qquad f_{yy} = \frac{2x}{y^3}.$$

by Ito:
$$dZ_t = \frac{dX_t}{Y_t} \cancel{+} - \frac{X_t}{Y_t}\frac{dY_t}{Y_t} + \frac{X_t}{Y_t}\frac{d[Y,Y]_t}{Y_t^2} \cancel{+} - \frac{d[X,Y]_t}{Y^2}$$

$$= \frac{X_t}{Y_t}\left(\mu_1 dt + \sigma_1 dW_{1,t}\right) - \frac{X_t}{Y_t}\left(\mu_2 dt + \sigma_2 dW_{2,t}\right).$$

$$+ \frac{X_t}{Y_t} \sigma_2^{2} \, dt \quad - \quad \frac{\sigma_1 \sigma_2 X_t \cancel{Y_t} \rho \, dt}{Y_t^{\cancel{2}}}$$

$$= Z_t \left[\left(\mu_1 - \mu_2 + \sigma_2^2 - \sigma_1 \sigma_2 \rho \right) dt + \sigma_1 \, dW_{1,t} - \sigma_2 \, dW_{2,t} \right]$$

(6) Compute QV of Z. ($[dZ, dZ]_t$)

$$[dZ, dZ]_t = [Z_t((\dots)dt + \sigma_1 dW_{1,t} - \sigma_2 dW_{2,t}, \; Z_t(\dots)dt + \sigma_1 dW_{1,t} - \sigma_2 dW_{2,t}]$$

$$\overset{\text{bilinearity}}{=} Z_t^2 [\sigma_1 dW_{1,t} - \sigma_2 dW_{2,t}, \; \sigma_1 dW_{1,t} - \sigma_2 dW_{2,t}]$$

$$= Z_t^2 \left([\sigma_1 dW_{1,t}, \; \sigma_1 dW_{1,t} - \sigma_2 dW_{2,t}] - [\sigma_2 dW_{2,t}, \; \sigma_1 dW_{1,t} - \sigma_2 dW_{2,t}] \right.$$

$$= Z_t^2 \Big([\sigma_1 dW_{1,t}, \sigma_1 dW_{1,t}] - [\sigma_1 dW_{1,t}, \sigma_2 dW_{2,t}] - [\sigma_2 dW_{2,t}, \sigma_1 dW_{1,t}] + [\sigma_2 dW_{2,t}, \sigma_2 dW_{2,t}] \Big)$$

$$= Z_t^2 \left(\underbrace{\sigma_1^2 dt - 2\sigma_1\sigma_2 \rho dt + \sigma_2^2 dt}_{\sigma_*} \right).$$

σ_* from the Magrabe sol'n.

↳ called instantaneous volatility w.r.t X and Y.

If $$dX_t = b_t dt + \sum_{i=1}^{d} \sigma_i(t) dW_i(t).$$

$$d[X, X] = d\left[\cancel{b_t dt} + \sum_{i=1}^{d} \sigma_i(t) dW_i(t), \cancel{b_t dt} + \sum_{i=1}^{d} \sigma_i(t) dW_i(t) \right].$$

$$= d\left[\sum_{i=1}^{d} \sigma_i(t) dW_{i,t}, \sum_{i=1}^{d} \sigma_i(t) dW_{i,t} \right].$$

$$= \sum_{i=1}^{d} \left[\sigma_i(t) dW_{i,t}, \sum_{j=1}^{d} \sigma_j(t) dW_{j,t} \right].$$

$$= \sum_{i=1}^{d} \sum_{j=1}^{d} \left[\sigma_i(t) dW_{i,t}, \sigma_j(t) dW_{j,t} \right].$$

$$= \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_i(t)\, \sigma_j(t)\, d[W_{i,t}, W_{j,t}].$$

still holds if $X_t = b_t\,dt + \sum_{i=1}^{d} \sigma_i(t)\, \cancel{dW_{i,t}}\, dY_{i,t}$.

↳ Ito process

$$d[X_t, X_t] = \sum_{i=1}^{d} \sum_{j=1}^{d} \cancel{\sigma_{ij}}\, \sigma_i(t)\, \sigma_j(t)\, d[Y_{i,t}, Y_{j,t}]$$

Change of Measure:

$\mathbb{P}$ prob measure. Z RV s.t.

(i) $Z > 0$ a.s and (ii) $\mathbb{E}[Z] = 1$

Then we define a <u>new</u> prob. measure $\tilde{\mathbb{P}}$.

where if A is an event then

$(\tilde{\mathbb{P}}(\Omega) = \mathbb{E}[Z] \overset{\text{need}}{=} 1)$

$$\tilde{\mathbb{P}}(A) := \mathbb{E}[Z \, \mathbb{1}_A]$$

we say $\tilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$ if.

$\mathbb{P}(A) = 0 \iff \tilde{\mathbb{P}}(A) = 0.$ and $\tilde{\mathbb{P}}(A) = 1 \iff \mathbb{P}(A) = 1.$

Ex Suppose $X \sim N(0,1)$, under $\mathbb{P}$.

define $d\tilde{\mathbb{P}} = \underbrace{e^{\alpha X + \beta}}_{Z} \, d\mathbb{P}$.

$\alpha \in \mathbb{R}$ is fixed. Find $\beta, \gamma \in \mathbb{R}$ s.t.

$X + \gamma \sim N(0,1)$ under $\tilde{\mathbb{P}}$.

sol'n. $Z > 0$ ✓.

we need $\mathbb{E}[Z] = 1$.

$\mathbb{E}[Z] = e^{\beta}\mathbb{E}[e^{\alpha X}] = e^{\beta + \frac{\alpha^2}{2}}$

So $\mathbb{E}[Z] = 1 \iff \beta = -\frac{\alpha^2}{2}$.

under $\mathbb{P}$ $X+\gamma \sim N(\gamma,1)$

We need $X+\gamma \sim N(0,1)$ under $\tilde{\mathbb{P}}$.

$$\tilde{\mathbb{P}}(\underbrace{X+\gamma}_{N(0,1)} \le \overset{\sim N(x)}{x}) = \tilde{\mathbb{P}}(X \le x-\gamma)$$

$$= \cancel{\int_{-\infty}^{x-\gamma} f_X(y)\,dx}\ \int_{\{X \le x-\gamma\}} \cancel{Z}\, d\tilde{\mathbb{P}}.$$

$$= \int_{\{X \le x-\gamma\}} Z\, d\mathbb{P} = \int_{-\infty}^{x-\gamma} Z\, f_X(y)\, dy.$$

$$= \int_{-\infty}^{x-\gamma} \left(e^{\alpha y - \frac{\alpha^2}{2}}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\right) dy.$$

$$= \int_{-\infty}^{x-\gamma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y^2 - 2\alpha y + \alpha^2)}\, dy.$$

$$= \int_{-\infty}^{x-\gamma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(y-\alpha)^2} dy.$$

Let $z = y - \alpha$. $dz = dy$.

$y \downarrow -\infty$, $z \downarrow -\infty$. $y \to x-\gamma$, $z \to x-\gamma-\alpha$.

$$= \int_{-\infty}^{x-\gamma-\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = N(x-\gamma-\alpha).$$

We seek that $X+\gamma \sim N(0,1)$ under $\tilde{\mathbb{P}}$

$\Longleftrightarrow \gamma = -\alpha$.

$\therefore X - \alpha \sim N(0,1)$ under $\tilde{\mathbb{P}}$

$Z = e^{\alpha X - \frac{1}{2}\alpha^2}$

<u>If</u> $\mathbb{E}[Z_t] = 1$ then.

$\tilde{W}_t$ is a BM under $\tilde{\mathbb{P}}$.

where $d\tilde{\mathbb{P}} = Z_t \, d\mathbb{P}$.

i.e. if X_t is an adapted process then

$$\tilde{\mathbb{E}}[X_t] = \mathbb{E}[X_t Z_t]$$

<u>Comment</u> if $\mathbb{E}[Z_t] = 1 \ \forall \ 0 \leq t \leq T$.

then $\tilde{W}_t$ is a BM for $t \in [0, T]$
under $\tilde{\mathbb{P}}$.

$$\mathbb{P}(X \le x - \gamma) = N(x-\gamma)$$

$$= \int_{-\infty}^{x-\gamma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\, dy.$$

Girsanov:

$$b = (b_1, \dots, b_d)$$

$$W = (W_1, \dots, W_d) \sim \text{standard BM.}$$

$$Z_t = e^{-\int_0^t b(s)\cdot dW_s - \frac{1}{2}\int_0^t |b(s)|^2 ds}.$$

$$\int_0^t b(s)\cdot dW_s = \sum_{i=1}^d b(s)\, dW_{i,s}.$$

$$|b(s)|^2 = \sum_{i=1}^d b_i(s)^2.$$

$$\tilde{W}_t = W_t + \int_0^t b(s)\, ds.$$

if $\quad \alpha = -\int_0^t \theta(s)\, dW_s$ "

and "$\frac{1}{2}\alpha^2 = \frac{1}{2}\int_0^t |\theta(s)|^2 ds$" then this

looks similar to our previous calculation.

<u>Ex</u>: $\quad dX_t = \gamma\, dt + 2\sqrt{X_t}\, dW_t$. $\rightarrow$ similar processes appear in the Heston Model.

define $\quad Z_t = e^{-\frac{\alpha}{2}\int_0^t \sqrt{X_s}\, dW_s - \frac{\alpha^2}{8}\int_0^t X_s\, ds}$.

You are told $\mathbb{E}[Z_t] = 1$.

Find dX_t under $\tilde{\mathbb{P}}$.

sol'n

Notice $\theta_t = -\frac{\alpha}{2}\sqrt{X_t}$. $(d=1)$.

so. $dX_t = \gamma\, dt + 2\sqrt{X_t}\,(\underbrace{dW_t + \theta_t\, dt}_{\tilde{W}} - \theta_t\, dt)$.

$$= (\gamma - 2\sqrt{X_t}\,\theta_t \cancel{dt})dt + 2\sqrt{X_t}\, d\tilde{W}_t$$

$$= (\gamma + \alpha \cdot X_t)dt + 2\sqrt{X_t}\, d\tilde{W}_t.$$

Risk - Neutral Pricing.

Black - Scholes Model:

European option with payoff ~~$V(S_T, T)$~~ V_T.

then time t price is.

$$V_t = \tilde{\mathbb{E}}[e^{-r(T-t)} V_T \mid \mathcal{F}_t]$$

RNM. $\frac{\mu_t - r_t}{\sigma_t}$

$S \sim GBM(\mu, \sigma)$. under $\mathbb{P}$

then $S \sim GBM(r, \sigma)$ under $\tilde{\mathbb{P}}$.

If $r = r_t$ can vary over time then.

$$V_t = \tilde{\mathbb{E}}[e^{-\int_t^T r_s ds} V_T \mid \mathcal{F}_t]$$

Ex: Forward Start option: B-S model.

Fix t_1, T with $0 < t_1 < T$. the payoff of our option at time T is

$$(S_T - S_{t_1})^+$$

↳ strike decided at time t_1.

but option entered into today.

Note: Cannot just use B-S call formula to compute directly time 0 price. since S_{t_1} is unknown today.

Note: typically we try to use independence to "lose" the conditional expectation in the RNP formula:

Exam Tips.

1). Know your formulas and how to use them

↳ Ito (multidim).

↳ Girsanov

↳ RNP.

↳ Levy's ~~Ch~~ criteria for BM.

↳ Ito Isometry

etc...

Regular Call.

$$C_{t_1} = \tilde{\mathbb{E}}\left[e^{-r(T-t_1)}\left(S_T - K\right)^+ \mid \mathcal{F}_{t_1}\right].$$

$$= S_{t_1}\,\tilde{\mathbb{E}}\left[e^{-r(T-t_1)}\left(\frac{S_T}{S_{t_1}} - \frac{K}{S_{t_1}}\right)^+\right]$$

Back to this calculation.

$$= S_0\, C(T-t_1, 1, 1)$$

$\underbrace{C(}_{\text{time remaining}}$ $T-t_1$, 1 — initial price of stock, 1 — strike price.

$$= C(T-t_1, S_0, S_0).$$

OR course from ⑥ you could use lazy statistician to get it, but this is lengthy.

We will compute time 0 price.

<u>sol'n.</u>

$$V_0 = \tilde{\mathbb{E}}\left[e^{-rT}(S_T - S_{t_1})^+\right].$$

$$= \tilde{\mathbb{E}}\left[e^{-rT} S_{t_1}\left(\frac{S_T}{S_{t_1}} - 1\right)^+\right].$$

$$\frac{S_T}{S_{t_1}} = e^{\sigma(\tilde{W}_T - \tilde{W}_{t_1}) + (r - \frac{1}{2}\sigma^2)(T - t_1)}.$$

$$\perp\!\!\!\perp \quad S_{t_1}.$$

$$= \tilde{\mathbb{E}}\left[e^{-rt_1} S_{t_1}\right] \tilde{\mathbb{E}}\left[e^{-r(T-t_1)}\left(\frac{S_T}{S_{t_1}} - 1\right)^+\right] \ (*).$$

$$= \underbrace{S_0}_{\text{mart.}} \tilde{\mathbb{E}}\left[e^{-r(T-t_1)}\left(e^{\sigma \tilde{W}_T - \tilde{W}_{t_1} + (r - \frac{1}{2}\sigma^2)(T-t)} - 1\right)^+\right]$$

*

2). Think about the answers you get and if they make sense.

e.g. If you are pricing an option and you get a price that can be negative or that it depends on future knowledge then clearly it is incorrect.

We understand that it may be hard to find your mistake in an exam, but if you write down ~~fr. cour~~ that you know the answer cannot be correct, you are more likely to get partial credit.

people got

e.g. $\mathbb{E}[X_t^2] < 0$ on Q1 midterm, which is an avoidable mistake.