Girsanov:

P (old measure).

$Z \longrightarrow$ RV. Define $\tilde{P}(A) = E(\mathbb{1}_A Z) = \int_A Z\, dP$

(new measure)

$$(d\tilde{P} = Z\, dP,$$

$$\tilde{E} X = E\, Z X$$

$\uparrow$ new measure $\qquad \uparrow$ old measure.

$$d\tilde{W}_i = b_i\, dt + dW_i \qquad (W \text{ is BM wrt } P).$$

Fix $T > 0$, $Z(T) = \exp\left(-\int_0^T b(s) \cdot dW(s) - \frac{1}{2}\int_0^T |b(s)|^2 ds\right).$

$$\left.\begin{array}{l}(dZ = -Z\,\theta \cdot dW) \\ \text{Let } d\tilde{P} = Z(T)\; dP\end{array}\right\} \Rightarrow \tilde{W} \text{ is a BM under } \tilde{P} \text{ (up to time } T).$$

Risk Neutral Pricing.

$S \to$ stock price.

$$dS(t) = \alpha(t) S(t)\,dt + \sigma(t) S(t)\,dW(t).$$

(mean return & volatility can depend on t).

Money Market: Variable interest rate $R(t)$.

($R \to$ adapted process).

Discount Process: $D(t) = \exp\left(-\int_0^t R(s)\,ds\right)$.

$$dD(t) = -R(t)\,D(t)\,dt.$$

<u>Def</u>: A <u>Risk Neutral Measure</u> is a measure $\tilde{P}$ ($\tilde{P}$ equiv to P). under which $D(t)\,S(t)$ is a mg.

<u>Remk</u>: ① Existence of RNM $\iff$ No arbitrage.

& ② Uniqueness of RNM $\iff$ (No arbitrage & all derivative securities can be hedged).

$\Big\}$ FTAP

Compute RNM:

$$d\big(D(t)S(t)\big) = D(t)\,dS(t) + S(t)\,dD(t).$$

$$= D\big(\alpha(t)S(t)\,dt + \sigma(t)S(t)\,dW(t)\big).$$

$$- R(t)S(t)D(t)\,dt.$$

$$= D(t)S(t)\sigma(t)\left(dW(t) + \frac{\alpha(t)-R(t)}{\sigma(t)}\,dt\right).$$

$$\theta(t) = \frac{\alpha(t)-R(t)}{\sigma(t)} \qquad \text{(market price of risk)}.$$

$$\Rightarrow d(DS) = DS\sigma\underbrace{\left(\theta(t)\,dt + dW(t)\right)}_{d\widetilde{W}}.$$

Let $\widetilde{W}(t) = \int_0^t \theta(s)\,ds + W(t)$.

Girsanov: $d\widetilde{P} = Z(T)\,dP$

$$Z(T) = \exp\left(-\int_0^T \theta(t)\,dW(t) - \frac{1}{2}\int_0^T \theta(t)^2\,dt\right).$$

Under $\widetilde{P}$, $\widetilde{W}$ is a BM $\Rightarrow$. $D(t)S(t)$ is a mg!

<u>Theorem</u> (Risk Neutral Price formula).

Let $V(T)$ = some $\mathcal{F}_T$ meas R.V.

(Represents payoff of a security).

$\tilde{P}$ = RNM. ($D(t)\,S(t)$ is mg under $\tilde{P}$).

The arbitrage free price of this security at $t \leq T$.

is given by

$$\boxed{V(t) = \tilde{E}\left(V(T) \exp\left(-\int_t^T R(s)\, ds \right) \Big| \, \mathcal{F}_t \right)}.$$

IOU: Pf

~~Pk~~ Compute dS:

$$dS = \alpha S\,dt + \sigma S\,dW$$

$$= \alpha S\,dt + \sigma S\left(\cancel{\alpha dt} - \theta\,dt + d\tilde{W}\right).$$

$$= \alpha S\,dt + S(R-\alpha)\,dt + \sigma S\,d\tilde{W}.$$

$$\boxed{dS = RS\,dt + \sigma S\,d\tilde{W}}$$

<u>Lemma</u>: $\Delta(t)$ = any adapted process.

$X(t)$ = value of Pf $\begin{cases} \to \Delta(t) \text{ shares in stock} \\ \to X(t) - \Delta(t)S(t) \text{ in cash} \end{cases}$

Then $D(t)X(t)$ is a mg under $\tilde{P}$.

Pf: Know $dX = \Delta(t)\, dS(t) + R(t)\left(X(t) - \Delta(t) S(t)\right) dt.$

$$\Rightarrow dX = \Delta\left(\cancel{RS\,dt} + \sigma S\,d\tilde{W}\right) + R\left(X - \cancel{\Delta S}\right) dt$$

$$= \underbrace{\Delta \sigma S\, d\tilde{W}}_{\tilde{P}\text{ mg}} + RX\, dt$$

Compute $d(DX) = D\,dX + X\,dD + \overbrace{d[D,X]}^{0}$

$$= \left(\Delta \sigma S\, d\tilde{W} + RX\,dt\right) D - RXD\,dt$$

$$= D\Delta S \sigma\, d\tilde{W} + 0 \Rightarrow DX \text{ is a } \tilde{P} \text{ mg.}$$

Proof of RNP formula:

$V(T) \longrightarrow$ payoff.

Let $X(t)$ = value of R. Pf at time t.

Know $X(T) = V(T)$. & DX is $\tilde{P}$ mg.

$$\Rightarrow X(t) = \frac{D(t)\,X(t)}{D(t)} = \frac{\tilde{E}(D(T)X(T) \mid \mathcal{F}_t)}{D(t)}.$$

$$= \tilde{E}\left(\frac{D(T)}{D(t)} V(T) \,\Big|\, \mathcal{F}_t \right).$$

$$= \tilde{E}\left(\exp\left(-\int_t^T R(s)\,ds\right) V(T) \,\Big|\, \mathcal{F}_t \right). \quad //$$

Remark: Delta Hedging.

Suppose $V(T) = f(S(T))$

$$\boxed{\text{Markov property: } V(t) = c(t, S(t))}$$

Equate $c(t, S(t)) = X(t)$ (wealth of R. Pf).

$d(\quad) = d(\,)$

$$\boxed{\longrightarrow \Delta(t) = \partial_x c(t, S(t))}.$$ ← Delta Hedging.

If $V(T)$ is not in the form $f(S(T))$ $\left(\text{Eg: } \frac{1}{T}\int_0^T S(t)\,dt\right)$. need not have Δ hedging.

Derive Black Scholes using RNP.

$$\left.\begin{aligned} \alpha(t) &= \alpha \quad (\text{const}). \\ \sigma(t) &= \sigma \quad (\text{const}). \\ R(t) &= r \quad (\text{const}). \end{aligned}\right\} \quad S = GBM(\alpha, \sigma).$$

$c(t, S(t)) =$ AFP of European call, (mat T, strike K).
at time t,

$$\text{RNP}: \ c(t, S(t)) = \tilde{E}\left(e^{-r(T-t)} (S(T) - K)^+ \,\middle|\, \mathcal{F}_t \right).$$

$$\boxed{\tau = T - t}$$

To compute: Observe under $\tilde{P}$, $dS = rS\,dt + \sigma S\,d\tilde{W}$

$\Rightarrow$ under $\tilde{P}$, S is $GBM(r, \sigma)$

<u>Formula:</u> $S(t) = S(0)\exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma\tilde{W}(t)\right).$

$$S(T) = S(0)\exp\left(\left(r - \frac{\sigma^2}{2}\right)T + \sigma\tilde{W}(T)\right).$$

$$\Rightarrow S(T) = S(t)\exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sqrt{\tau}\,\sigma\underbrace{\frac{\tilde{W}(T) - \tilde{W}(t)}{\sqrt{\tau}}}_{\sim N(0,1)}\right).$$

$$\Rightarrow c(t, S(t)) = e^{-r\tau}\tilde{E}\left[\left(S(t)\exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\left(\frac{\tilde{W}(T) - \tilde{W}(t)}{\tau}\right)\right) - K\right)^+ \Big| \mathcal{F}_t\right]$$

$$\circledast \overset{\text{Ind Lemma}}{=} e^{-r\tau}\int_{y\in\mathbb{R}} \left(S(t)\exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\, y\right) - K\right)^+ \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy.$$

Simplify:

$$x = S(t).$$

Q: $x\exp\left(\left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\, y\right) = K$

$$\Leftrightarrow \left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\, y = \ln\left(\frac{K}{x}\right).$$

$$\Leftrightarrow y = \frac{1}{\sigma\sqrt{\tau}}\left(-\left(r-\frac{\sigma^2}{2}\right)\tau + \ln\left(\frac{K}{x}\right)\right).$$

$$= \frac{-1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r-\frac{\sigma^2}{2}\right)\tau\right).$$

$$\stackrel{\text{def}}{=} -d_-(x).$$

$$\Rightarrow \text{from } \circledast : c(t,x) = e^{-r\tau}\int_{y=-d_-(x)}^{\infty}\left(x\exp\left(\left(r-\frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\,y\right) - K\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}\,dy.$$

$$= ① - ②$$

(2): $e^{-r\tau} \int_{y=-d_-(x)}^{\infty} K e^{-y^2/2} \frac{dy}{\sqrt{2\pi}}$.

$$= e^{-r\tau} \int_{-\infty}^{d_-(x)} K \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy,$$

$$= K e^{-r\tau} N\left(d_-(x)\right).$$

CDF of $N(0,1)$.

①: $$e^{-r\tau}\int_{-d_-}^{\infty} x \exp\left(\left(r-\frac{\sigma^2}{2}\right)\tau + \sigma\sqrt{\tau}\,y - \frac{y^2}{2}\right)\frac{dy}{\sqrt{2\pi}}$$

$$= \int_{-d_-}^{\infty} x \exp\left(-\frac{\sigma^2}{2}\tau + \sigma\sqrt{\tau}\,y - \frac{y^2}{2}\right)\frac{dy}{\sqrt{2\pi}}$$

$$= \int_{-d_-}^{\infty} x \exp\left(-\frac{(y-\sigma\sqrt{\tau})^2}{2}\right)\frac{dy}{\sqrt{2\pi}}$$

$(z = y - \sigma\sqrt{\tau}$

$$= \int_{-(d_- + \sigma\sqrt{\tau})}^{\infty} x \exp\left(-\frac{z^2}{2}\right)\frac{dz}{\sqrt{2\pi}}$$

$$= x \int_{-\infty}^{(d_- + \sigma\sqrt{\tau})} e^{-z^2/2} \frac{dz}{\sqrt{2\pi}}$$

$$= x \, N\Big(\underbrace{d_-(x) + \sigma\sqrt{\tau}}_{d_+} \Big).$$

$$d_+(x) = d_-(x) + \sigma\sqrt{\tau}$$

$$= \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau \right) + \sigma\sqrt{\tau}$$

$$= \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\tau \right), \quad \cancel{z} \quad //.$$

$$\Rightarrow c(t,x) = \text{1} - \text{2}$$

$$= x\,N(d_+) - K e^{-r\tau} N(d_-) \,. \quad //.$$