Risk Neutral Measures.

__Motivation:__ $S \longrightarrow \cancel{GBM}\ GBM(\alpha, \sigma)$ (Stock Price).

Constant Interest rate r.

$$X(t) = \text{any Pf} = \begin{cases} \Delta(t) & \text{shares in stock.} \\ X(t) - S(t)\Delta(t) & \text{in cash.} \end{cases}$$

$\boxed{\text{IF}}$ $\underline{\alpha = r}$. then: $e^{-rt} X(t)$ is a mg !! ($\cancel{\text{IOU}}$).
(You can check Itô).

~~Say~~ Consider a security on S which pays $V(T)$ at maturity T.

Let $X(t)$ = value of R. Pf.

$\Rightarrow X(T) = V(T)$. & $e^{-rt}X(t)$ is a mg.

Price security: $X(t) = e^{rt}\left(\underbrace{e^{-rt}X(t)}\right)$.

$$= e^{rt}\left(E\left(e^{-rT}V(T) \mid \mathcal{F}_t\right)\right).$$

$$\boxed{\text{Price at time } t = E\left(e^{-r(T-t)}V(T) \mid \mathcal{F}_t\right)}.$$

$\boxed{\text{Goal}}$: Replace $P \longrightarrow$ new measure $\tilde{P}$ such that

I.e. under $\tilde{P}$, mean return rate (S) = Interest rate.

① Change of measure:

$A \subseteq \Omega$ some event.

$P(A) \longrightarrow$ prob of A occuring.

Define $\tilde{P}(A) \longrightarrow$ prob of A occuring (under a new measure $\tilde{P}$).

Typically: Set $\tilde{P}(A) = E(Z \mathbb{1}_A) = \int_A Z \, dP$.

$Z \rightarrow$ some R.V.

Need ① $Z \geq 0$ (in fact $Z > 0$).

② $\tilde{P}(\Omega) = 1 \iff EZ = 1$.

<u>Notation:</u> If $\tilde{P}(A) = \int_A Z \, dP$, we say $\frac{d\tilde{P}}{dP} = Z$.

(Radon Nikodym derivative of $\tilde{P}$ wrt P).

<u>Claim:</u> X is a R.V. EX = expected value of X wrt the measure P.

$\tilde{E}X$ = expected value of X ~~wrt~~ wrt the new measure $\tilde{P}$.

$$\tilde{E}X = \int X \, d\tilde{P} = \int XZ \, dP = E(XZ).$$

$$\boxed{\tilde{E}X = E(XZ)}$$

Notation: $\tilde{E}(X \mid \mathcal{G})$ = cond exp of X given $\mathcal{G}$ wrt $\tilde{P}$.

Theorem (the Cameron–Martin–Girsanov theorem).

Let $b(t) = (b_1(t), \dots b_d(t))$. d-dim adapted proc.

$W(t) = (W_1(t), \dots W_d(t))$ " " B.M.

Let $$\tilde{W}(t) = W(t) + \int_0^t b(s)\, ds.$$

$$\left(\text{i.e. } \tilde{W}_i(t) = W_i(t) + \int_0^t b_i(s)\, ds\right).$$

Let $$Z(t) = \exp\left(- \int_0^t b(s) \cdot dW(s) \cancel{+} - \frac{1}{2} \int_0^t |b(s)|^2 \, ds \right).$$

$$\left(\int_0^t b(s) \cdot dW(s) = \sum_i \int_0^t b_i(s) \, dW_i(s) \right.$$

$$\left. \& \quad |b(s)|^2 = \sum_i b_i(s)^2 \right).$$

Fix $T > 0$. Define $\tilde{P}$ by $\boxed{d\tilde{P} = Z(T) \, dP}$

$$\left(\text{i.e. } \tilde{P}(A) = \int_A Z(T) \, dP \right).$$

<u>If</u> Z is a mg, then $\tilde{W}$ is a BM under $\tilde{P}$ (up to time T).

Note: $Z(0) = 1$. & $Z > 0$.

If Z is a mg $E(Z(T)) = E(Z(0)) = 1$

$\Rightarrow \tilde{P}$ is a prob measure.

Compute dZ.

Let $M(t) = \int_0^t b(s) \cdot dW(s) = \sum_{i=1}^d \int_0^t b_i(s)\, dW_i(s)$.

Let $\cancel{f(x,t)}\ f(t,x) = \exp\left(-x - \frac{1}{2}\int_0^t |b(s)|^2 ds\right)$.

$\Rightarrow Z(t) = \cancel{f(M)}\ f(t, M(t))$.

$$\Rightarrow \quad dZ = \partial_t f \, dt + \partial_x f \, dM + \frac{1}{2} \partial_x^2 f \, d[M,M].$$

① $$\partial_t f(t,x) = \cancel{f}\left(-\frac{1}{2}|b(t)|^2\right) . f .$$

② $$\partial_x f = -f \quad \& \quad \partial_x^2 f = f$$

③ $$d[M,M] = \sum_{j,i=1}^{d} b_i \, b_j \, d[W_i, W_j] = \sum b_i^2 \, dt$$
$$= |b|^2 \, dt.$$

$$\Rightarrow dZ = -Z \, dM + \left(f\cancel{\left(-\frac{1}{2}|b(t)|^2\right)} + \frac{f}{\cancel{2}} \cancel{|b|^2} \right) dt$$

$$\boxed{\Rightarrow dZ = -\sum_{i=1}^{d} Z(t)\, b_i(t)\, dW_i} + 0$$

$\overset{?}{\Longrightarrow}$ Z is a mg

(NO! Z is only guaranteed to be a mg if

$$E\int_0^t Z(s)^2 |b(s)|^2 ds < \infty,$$

Need not be true in general!),

Idea of Pf:

Use Levy: ① Check $\tilde{W}$ to be cts mg under $\tilde{P}$.

② $d[\tilde{W}_i, \tilde{W}_j] = \mathbb{1}_{\{i=j\}} dt$.

Note ② Easy: $d[\tilde{W}_i, \tilde{W}_j] = d[W_i, W_j]$.

$= \mathbb{1}_{\{i=j\}} dt$.

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① $\rightarrow$ Will check now.

Formula for Cond exp under $\tilde{E}$:

Let $0 \leq s \leq t \leq T$.

$X \longrightarrow \mathcal{F}_t$ measurable.

$$\boxed{\tilde{E}(X \mid \mathcal{F}_s) = \frac{1}{Z(s)} E(Z(t)\, X \mid \mathcal{F}_s).}$$

new measure

old measure.

<u>Proof</u>: ① Pick $A \in \mathcal{F}_s$.

Compute $\int_A \tilde{E}(X \mid \mathcal{F}_s)\, d\tilde{P}$ in 2 ways.

① $$\int_A \tilde{E}(X \mid \mathcal{F}_s)\, d\tilde{P} \overset{\text{def of } \tilde{P}}{=\!=} \int_A \tilde{E}(X \mid \mathcal{F}_s)\, Z(T)\, dP.$$

② $$\int_A \tilde{E}(X \mid \mathcal{F}_s)\, d\tilde{P} \overset{\text{def of } \tilde{E}}{=\!=} \int_A X \, d\tilde{P}$$

Simplify both: ② : $$\int_A X \, d\tilde{P} = \int_A X \, Z(T)\, dP.$$

$$= \int_A \cancel{X}\, E\big(X Z(T) \mid \mathcal{F}_s\big)\, dP.$$

$$= \int_A E\Big(E\big(X Z(T) \mid \mathcal{F}_t\big) \,\Big|\, \mathcal{F}_s\Big)\, dP.$$

$$= \int_A E\big(X \cancel{(t)}\, Z(t) \mid \mathcal{F}_s\big)\, dP$$

① $\therefore$ $\displaystyle\int_A \tilde{E}(X \mid \mathcal{F}_s)\, d\tilde{P} = \int_A \tilde{E}(X \mid \mathcal{F}_s)\, Z(T)\, dP$

$$= \int_A E\left(\tilde{E}(X \mid \mathcal{F}_s)\, Z(T) \,\middle|\, \mathcal{F}_s \right) dP.$$

$$= \int_A \tilde{E}(X \mid \mathcal{F}_s)\, \underbrace{E\left(Z(T) \,\middle|\, \mathcal{F}_s \right)}_{Z(s)} dP$$

$\Rightarrow$ for every $A \in \mathcal{F}_s$.

$$\int_A E\left(X \cancel{(t)}\, Z(t) \,\middle|\, \mathcal{F}_s \right) dP = \int_A Z(s)\, \tilde{E}(X \mid \mathcal{F}_s)\, dP$$

$$\Rightarrow E\left(X Z(t) \mid \mathcal{F}_s\right) = Z(s)\tilde{E}(X \mid \mathcal{F}_s) \quad \text{a.s.}$$

$$\boxed{\Rightarrow \tilde{E}(X \mid \mathcal{F}_s) = \frac{1}{Z(s)} E\left(X Z(t) \mid \mathcal{F}_s\right).}$$

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Lemma: An adapted process $M(t)$ is a mg wrt $\tilde{P}$ ← new

$\Longleftrightarrow$ $M(t)Z(t)$ is a mg wrt $\cancel{\tilde{P}}$ P ← old.

Pf: Say M is a mg under $\tilde{P}$.

$$\Rightarrow \tilde{E}(M(t) \mid \mathcal{F}_s) = M(s).$$

$$\Rightarrow M(s) = \frac{1}{Z(s)} E\left(M(t) Z(t) \mid \mathcal{F}_s\right)$$

$$\Rightarrow M(s)Z(s) = E\left(M(t)Z(t) \mid \mathcal{F}_s\right)$$

$\Rightarrow$ MZ is a mg under P.

<u>Converse</u>: Suppose MZ is a mg under P.

$$\Rightarrow M(s)Z(s) = E\left(M(t)Z(t) \mid \mathcal{F}_s\right).$$

$$\Rightarrow M(s) = \frac{1}{Z(s)} E\left(M(t)Z(t) \mid \mathcal{F}_s\right) = \tilde{E}\left(M(t) \mid \mathcal{F}_s\right).$$

$\Rightarrow$ M is a mg under $\tilde{P}$!

Proof of Girsanov:

$$d\tilde{W}_i = dW_i + b_i \, dt.$$

Z is a mg. $dZ = -\sum_{j=1}^{d} Z(t) b_j(t) \, dW_j$

$d\tilde{P} = Z(T) \, dP.$

<u>Claim:</u> $\tilde{W}$ is a BM under $\tilde{P}$.

<u>Proof:</u> Know $\tilde{W}$ is cts & $d[\tilde{W}_i, \tilde{W}_j] = 1_{\{i=j\}} dt.$

Only need to show $\tilde{W}_i$ is a mg under $\tilde{P}$.

$\Longleftrightarrow$ Showing $\tilde{W}_i Z$ is a mg under P...

$\tilde{W}_i(t)\, Z(t)$

<u>Check:</u> $d(\tilde{W}_i Z) = \tilde{W}_i\, dZ + Z\, d\tilde{W}_i + d[\tilde{W}_i, Z]$

$$= \underbrace{\tilde{W}_i\, dZ}_{mg} + Z(dW_i + b_i\, dt) + (-Z b_i)\, dt$$

$$= \underbrace{\tilde{W}_i\, dZ + Z\, dW_i}_{Mg} + 0$$

QED.