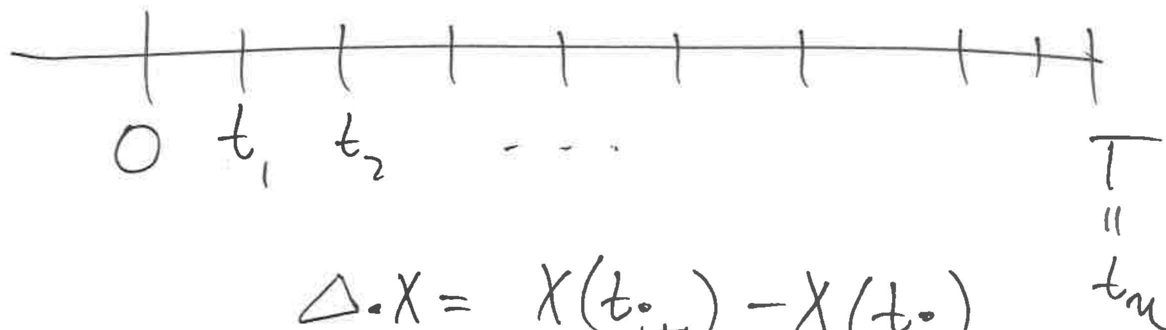
Last time: Multi Dim Itô formula.

↳ 1 Joint QV. X, Y 2 Itô processes.

$$[X, Y](T) = \lim_{\|P\| \to 0} \sum_i (\Delta_i X)(\Delta_i Y)$$



$$\Delta_i X = X(t_{i+1}) - X(t_i).$$

Thm (Itô): ① $X_1, X_2, \ldots X_n \to n$ stochastic processes.

$f = f(t, x_1, x_2, \ldots x_n)$ (non random fn).

↳ $C^{1,2}$: $\partial_t f, \partial_1 f, \ldots \partial_n f$

& $\partial_i \partial_j f$ all exist & are cts. ($i, j \in \{1, \ldots n\}$)

Let $X = (X_1, X_2, \ldots X_n)$ (n-dim process).

Itô:
$$f(T, X(T)) = f(0, X(0)) + \int_0^T \partial_t f(t, X(t))\, dt$$
$$+ \sum_{i=1}^n \int_0^T \partial_i f(t, X(t))\, dX_i(t)$$
$$+ \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n \int_0^T \partial_i \partial_j f(t, X(t))\, d[X_i, X_j](t)$$

Diff form: $$d\big(f(t, X(t))\big) = \partial_t f \, dt + \sum_{i=1}^{n} \partial_i f \, dX_i(t) + \frac{1}{2} \sum_{i,j=1}^{n} \partial_i \partial_j f \, d[X_i, X_j](t)$$

(Drop $(t, X(t))$ from above for brevity).

Intuition: $P = \{0 = t_0, t_1, \dots, t_m = T\}$.

Simplicity: $n = 2$. Processes X, Y & $f = f(x, y)$.

Let $\xi_i = (X(t_i), Y(t_i))$.

$$f(X(T),Y(T)) - f(X(0),Y(0)) = \sum_{i=0}^{m-1} f(\xi_{i+1}) - f(\xi_i).$$

$$f(\xi_{i+1}) - f(\xi_i) = \partial_x f(\xi_i) \cdot \Delta_i X + \partial_y f(\xi_i)\, \Delta_i Y$$

$$+ \frac{1}{2}\Big(\partial_x^2 f\, (\Delta_i X)^2 + \partial_y^2 f\, (\Delta_i Y)^2$$

$$+ 2\partial_x \partial_y f\, (\Delta_i X)(\Delta_i Y)\Big).$$

$$+ \text{ Higher order terms.}$$

① $\lim\limits_{\|P\| \to 0} \sum\limits_i \text{Higher order terms} = 0.$

② $$\lim_{\|P\|\to 0} \sum \partial_x f(\xi_i)\, \Delta_i X = \int_0^T \partial_x f(\ )\, dX(t)$$

$$\lim_{\|P\|\to 0} \sum \partial_y f(\xi_i)\, \Delta_i Y = \int_0^T \partial_y f\, dY$$

$$(\Delta_i X)^2 \approx [X, X](t_{i+1}) - [X, X](t_i).$$

$$= \Delta_i [X, X].$$

Expect $$\lim_{\|P\|\to 0} \sum_i \partial_x^2 f\, (\Delta_i X)^2 = \int_0^T \partial_x^2 f\, d[X, X](t)$$

$$\Rightarrow \lim_{\|P\|\to 0} \sum_i \partial_y^2 f\, (\Delta_i Y)^2 = \int_0^T \partial_y^2 f\, d[Y, Y](t).$$

$4ab = (a+b)^2 - (a-b)^2 \Rightarrow$ $$\lim_{\|P\|\to 0} 2\sum_i \partial_x \partial_y f (\Delta_i X)(\Delta_i Y) = 2\int_0^T \partial_x \partial_y f\, d[X, Y](t).$$

$$\Rightarrow f(X(T),Y(T)) - f(X(0),Y(0)) = \int_0^T \partial_x f \, dX + \int_0^T \partial_y f \, dY$$

$$+ \frac{1}{2}\left(\int_0^T \partial_x^2 f \, d[X,X] + \int_0^T \partial_y^2 f \, d[Y,Y] \right.$$

$$\left. + 2\int_0^T \partial_x \partial_y f \, d[X,Y] \right).$$

Joint Q.V.

① Recall M is a mg $\Rightarrow M^2 - [M,M]$ is a mg.

Also if A is any cts inc adapted process such that $A(0)=0$ & $M^2 - A$ is a mg $\Rightarrow A = [M,M]$.
(+ finiteness cond).

Joint QV:

Suppose M, N are 2 cts mg $(\{\mathcal{F}_t\})$.

& $E\,M(t)^2 < \infty$ & $E\,N(t)^2 < \infty$.

Then ① $MN - [M, N]$ is a mg.

& ② If A is any cts & adapted process with finite first variation & $A(0) = 0$.

& $MN - A$ is a mg $\Rightarrow A = [M, N]$.

Pf: ① $4MN = (M+N)^2 - (M-N)^2$.

$$4\left(MN - [M,N]\right) = \underbrace{(M+N)^2 - [M+N, M+N]}_{\text{mg}} - \underbrace{\left((M-N)^2 - [M-N, M-N]\right)}_{\text{mg}} \Bigg\} \text{ mg}.$$

Prop: (Bi-linearity). $X, Y, Z \longrightarrow$ Ito processes.

$\alpha \in \mathbb{R}$ (non random).

$$\Rightarrow [X, Y + \alpha Z] = [X, Y] + \alpha [X, Z].$$

Pf: $$[X, Y + \alpha Z] = \lim_{\|P\| \to 0} \sum \Delta_i(X) \left(\Delta_i (Y + \alpha Z) \right).$$

$$= \lim_{\|P\| \to 0} \sum (\Delta_i X) \left(\Delta_i Y + \alpha \Delta_i Z \right).$$

$$= [X, Y] + \alpha [X, Z].$$

Note: $[X+Y, X+Y] = [X, X] + [Y, Y] + 2[X, Y]$, Prop: $X_1, X_2 \to 2$ Itô processes.

$\sigma_1, \sigma_2 \to 2$ adapted processes.

$$I_1(t) = \int_0^t \sigma_1(s)\, dX_1(s) \quad \& \quad I_2(t) = \int_0^t \sigma_2(s)\, dX_2(s).$$

$$[I_1, I_2](t) = \int_0^t \sigma_1(s)\, \sigma_2(s)\, d[X_1, X_2](s).$$

$$\left(\text{Note } [I_1, I_1] = \int_0^t \sigma_1(s)^2\, d[X_1, X_1](s).\right)$$

Pf: $\cancel{I_i(T)}$ $i \in \{1, 2\}$.

$$I_i(T) = \lim_{\|P\| \to 0} \sum_j \sigma_i(t_j)\, \Delta_j X_i$$

$$[I_1, I_2] = \lim_{\|P\|\to 0} \sum_{i=0}^{n-1} (\Delta_i I_1)(\Delta_i I_2).$$

$$= \lim_{\|P\|\to 0} \sum_{i=0}^{n-1} \sigma_1(t_i)\, \Delta_i X_1 \quad \sigma_2(t_i)\, \Delta_i X_2.$$

$$= \lim_{\|P\|\to 0} \sum_{i=0}^{n-1} \sigma_1(t_i)\, \sigma_2(t_i)\, \underbrace{(\Delta_i X_1)(\Delta_i X_2)}_{\Delta_i [X_1, X_2]}.$$

$$= \int_0^T \sigma_1(t)\, \sigma_2(t)\, d[X_1, X_2].$$

Prop: M, N, 2 mg (M, N cts).

$$E\, M(t)^2 < \infty, \quad E\, N(t)^2 < \infty.$$

If M & N are Independent then $[M, N] = 0$

Proof: Know $MN - [M, N]$ is mg.

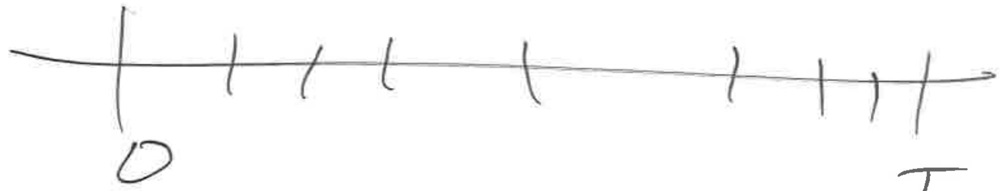
If we show MN is a mg $\Rightarrow [M, N] = 0$.

$$E\left(M(t)N(t) \mid \mathcal{F}_s\right) = E\left(M(t) \mid \mathcal{F}_s\right) E\left(N(t) \mid \mathcal{F}_s\right).$$

WRONG.

Know $E\, M(t)N(t) = E\, M(t)\, E\, N(t)$

Real Proof:



$$P = \{t_0 \cdots t_n = T\}.$$

NTS $\quad [M,N] = 0$

Show $\quad E[M,N]^2(T) = 0 \Rightarrow$

$$E\left(\sum (\Delta_i M)(\Delta_i N)\right)^2 = E\sum (\Delta_i M)^2 (\Delta_i N)^2$$

$$+ \underbrace{2E\sum_{j=0}^{n-1}\sum_{i=0}^{j-1} (\Delta_i M)(\Delta_i N)(\Delta_j M)\,\Delta_j N}_{\text{claim } = 0}.$$

$$E(\Delta_i M)(\Delta_i N)(\Delta_j M)(\Delta_j N) = \left[E\left((\Delta_i M)(\Delta_j M)\right)\right]\left[E(\Delta_i N)(\Delta_j N)\right].$$

$$= E\Big[\Delta_i M\, \underbrace{E\left(\Delta_j M \,\middle|\, \mathcal{F}_{t_j}\right)}_{\substack{0 \\ (M \text{ is a mg})}}\Big]\,\Big[\qquad\Big].$$

Here $E\left(\sum (\Delta_i M)^2 (\Delta_i N)^2\right) = E\sum (\Delta_i M)^2 (\Delta_i N)^2.$

$$\le E\Big(\underbrace{\max_i (\Delta_i N)^2}_{\xrightarrow[\|P\|\to 0]{} N \ (N \text{ is cts}),}\Big)\underbrace{\sum_i (\Delta_i M)^2}_{\xrightarrow[\|P\|\to 0]{} [M,M] = 0}.$$

Note: Converse is false.

$$[M,N]=0 \not\Longrightarrow M \ \& \ N \text{ are ind.}$$

Eg: $$M(t) = \int_0^t \mathbb{1}_{\{W(s)>0\}} \, dW(s).$$

$$N(t) = \int_0^t \mathbb{1}_{\{W(s)\leq 0\}} \, dW(s).$$

Q: $$[M,N](t) = \int \underbrace{\mathbb{1}_{\{W(s)>0\}} \mathbb{1}_{\{W(s)\leq 0\}}}_{0,} \, d[W,W]. = 0$$

$M+N = W$ (can check M & N are not ind.).

d dimensional BM:

$W = (W_1, W_2, \cdots W_d)$ is a standard d-dim

BM if ① Each coordinate is a 1 dim ~~B~~ std BM.

(Each W_i is a).

& ② If $i \neq j$, W_i is independent of W_j.

$$\text{QV: } d[W_i, W_j] = \begin{cases} 0 & i \neq j \\ dt & i = j \end{cases}$$

$$= \mathbb{1}_{\{i=j\}} \, dt.$$

Theorem: (Levy)

If $M = (M_1, M_2, \dots M_d)$ is a $\boxed{d\text{-dim mg.}}$

& M is $\boxed{\text{cts.}}$, $M(0) = 0$,

& $d[M_i, M_j] = \mathbb{1}_{\{i=j\}} \, dt.$

$\Longrightarrow$ M must be a std d-dim BM.

Review problem 7.5.

Compute the joint MGF of (M_i, M_j) using Itô.

Eg 1:

$$B(t) = \int_0^t \frac{W_1(s)}{|W(s)|}\, dW_1(s) + \int_0^t \frac{W_2(s)}{|W(s)|}\, dW_2(s).$$

$W \rightarrow$ 2 dim BM.

$W = (W_1, W_2)$.

$|W| = \sqrt{W_1^2 + W_2^2}$.

Claim: B is a BM.

Pf: ① B is cts ② B is a mg.

③ $$d[B,B] = \frac{W_1^2(t)}{|W(t)|^2}\,dt + \frac{W_2(t)^2}{|W(t)|^2}\,dt + 0$$
$$= dt$$

By Levy, B is a B.M.

Eg: $W \longrightarrow$ 2 dim B.M.

$$X(t) = \ln\left(|W(t)|^2\right) = \ln\left(W_1(t)^2 + W_2(t)^2\right).$$

Q: Is X a mg.

Sol: Itô. Compute dX & hope there are no dt terms.

$$f = f(t, x_1, x_2) = \ln\left(x_1^2 + x_2^2\right) = \ln\left(|x|^2\right).$$

$$\partial_t f = 0, \quad \partial_1 f = \frac{2x_1}{|x|^2}, \quad \partial_2 f = \frac{2x_2}{|x|^2}.$$

& Claim $\partial_1^2 f + \partial_2^2 f = 0$ (you check).

Ito: $$dX = \partial_t f\, dt + \partial_1 f\, dW_1 + \partial_2 f\, dW_2$$

$$+ \frac{1}{2}\Big[\partial_1^2 f\, dt + \partial_2^2 f\, dt + 2\,\partial_1\partial_2 f\, d\underbrace{[W_1, W_2]}_{0} \Big].$$

$$= \partial_1 f\, dW_1 + \partial_2 f\, dW_2 + \underbrace{(\partial_1^2 f + \partial_2^2 f)}_{0}\, dt.$$

$\Longrightarrow$ Guess X is a mg.

Claim: X is NOT a mg.

Compute $$E X(t) = \int_{\mathbb{R}^2} \ln|x|^2 \, e^{-|x|^2/2t} \frac{dx_1\, dx_2}{2\pi t}$$

Can check $\lim_{t \to \infty} EX(t) = +\infty$.

Q: What Went Wrong?