Black - Scholes Extensions.

- Dividends.
- pricing options on multiple underlyings.

(i). Original B-S assumes S_t does not pay dividends, but we want to allow for dividends.

Q: Do dividends even matter for option pricing?

A: If you hold a position in the asset you get dividend payments, but option holders do not! This will affect the replicating/hedging argument.

Recall B-S Assumptions.

1). S follows Geometric Brownian Motion.

2). Access to a risk-free Bank account

$$dB = rB\,dt$$

3). Can buy/sell fractions of shares.

4). Frictionless Market (no trans costs).

5) No arbtigrage in the market

We need to discuss how we want to model dividend payments.

We will assume S pays continuous dividends at rate q.

Think about it as follows: μ is the total return of S (i.e. with dividends)

Observe market price of S

$$dS_t = (\mu - q) S\,dt + \sigma S\,dW_t.$$

If I own the stock I receive

$$d\tilde{S}_t = \mu S\,dt + \sigma S\,dW_t.$$

We can think about it as follows.

$$\tilde{S}_t = S_t + D_t \qquad \text{where } dD_t = q S_t\,dt.$$

$\tilde{S}_t$ ← total package. return + dividends

S_t ← observable market price i.e. tradable price.

Remarks:

1). In reality dividends are paid out discretely (e.g quarterly).

2). Dividend schedules are not known exactly. We have to estimate them.

3). Imagine S is not a single stock, but an index. In this case continuous dividends is close enough.

4) We implicitly dividends are reinvested

Let's derive a PDE for a European option.

$V(S_t, t)$ be the value at time t of the option expiring at time T.

Replicating vs Hedging.

↳ create strategy to replicate option payoff.

↳ take position in option and stock and HAY to eliminate risk.

Let π be our portfolio.

$$\pi = V - \underbrace{\Delta}_{\text{to be chosen}} S + \underbrace{(\Delta S - V)}_{\text{BANK ACCOUNT}}$$

$$d\pi = dV - \Delta\, d\underbrace{\tilde{S}}_{(S+D)}$$

$$\overset{\text{Ito}}{=} V_t dt + V_S dS + \tfrac{1}{2} V_{SS} \sigma^2 S^2 dt - \Delta dS_t - \Delta dD_t$$

$$= (V_S - \Delta) dS + (V_t + \tfrac{1}{2} V_{SS} \sigma^2 S^2 - \Delta q S) dt$$

Choose $\Delta = V_S$.

$$d\pi = \left(V_t + \frac{1}{2} V_{SS}\, \sigma^2 S^2 - \overset{N-\Delta}{\Delta} q S\right)dt = r\pi\, dt$$

$$= r(V - V_S S)\, dt$$

$$\Rightarrow V_t + \overset{(r-q)}{\cancel{}}\, S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS} - rV = 0$$

$$V_t + (r-q) S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS} - rV = 0$$

THIS PDE IS A GLOBAL RULE

All European option satisfy this.

Contract specifications appear in BC's + TC's

Before we solve this let's get some intuition about dividends on a simpler asset.

Consider a prepaid forward:

Owner gets asset at time T for sure but pays now.

Let's price this in the case of dividends and no dividends.

Case 1: No dividends: S_t $t < T$.

This asset is a call with $K = 0$

You can check B-S formula.

here $\Delta = 1$.

Case 2: with dividends.

If I try to replicate by buying $\Delta = 1$ share

then at time T I will accumulate

$e^{q(T-t)}$ shares!

Not replicated, but easy fix:

→ tailed position.

If I buy $\Delta = e^{-q(T-t)}$ shares then at

time T I will have 1 share.

price will be $S_t e^{-q(T-t)}$

$dD = q S_t dt$ → gives exponential.

Now let's solve dividend B-S PDE in terms of "regular" B-S price.

Let $U(S,t)$ be price of option on non-dividend paying stock

i.e. U satisfies original B-S. $f(S,t)$.

We will show $V(S,t) = U(S_t e^{-q(T-t)}, t)$ then it will solve the dividend PDE.

Check: $V_t(S,t) = U_t(S_t e^{-q(T-t)}, t) + U_S(S_t e^{-q(T-t)}, t) \times q S_t e^{-q(T-t)}$

$V_S(S,t) = e^{-q(T-t)} U_S(S_t e^{-q(T-t)}, t)$

$V_{SS}(S,t) = e^{-2q(T-t)} U_{SS}(S_t e^{-q(T-t)}, t).$

Let $\bar{S}_t =: S_t e^{-q(T-t)}$

$$V_t + (r-q) S_t V_S + \frac{1}{2}\sigma^2 S^2 V_{SS} - rV$$

$$= u_t(\bar{S},t) + q\bar{S}\, u_s(\bar{S},t) + (r-q)\underbrace{S_t e^{-q(T-t)}}_{\bar{S}}\, u_s(\bar{S},t)$$

$$+ \frac{1}{2}\sigma^2 \underbrace{S^2 e^{-2q(T-t)}}_{(\bar{S})^2}\, u_{ss}(\bar{S},t) - r\,u(\bar{S},t)$$

$$= u_t(\bar{S},t) + \cancel{q\bar{S}\, u_s(\bar{S},t)} + (r\cancel{-q)\,\bar{S}\, u_s(\bar{S},t)}$$

$$+ \frac{1}{2}\sigma^2 (\bar{S})^2 u_{ss}(\bar{S},t) - r\,u(\bar{S},t) = 0$$

by the fact that

u solves B-S PDE.

Ex Call:

TC: $\cancel{E S_T}$ $C(S_T, T) = (S_T - K)^+$

BC: ① $C(0, T) = 0$

$$\cancel{\lim_{S\to\infty} \left(C(S,t) - (S_T e^{-\delta(T} - K) \right) = 0}$$

$$\lim_{S\to\infty} \left(C(S,t) - (S_t e^{-\delta(T-t)} - K e^{-r(T-t)}) \right) = 0.$$

$\Longleftrightarrow$ $\cancel{\frac{dC}{dS} : C_S(S,t)}$

② $\lim_{S\to\infty} C_S(S,t) = e^{-\delta(T-t)}.$

By our calculation price is

$$S_t e^{-q(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2).$$

$$d_1 = \frac{\ln\left(\frac{S_t e^{-q(T-t)}}{K}\right) - \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$\ln\left(\frac{S_t}{K} e^{-q(T-t)}\right) = \log\left(\frac{S_t}{K}\right) - q(T-t)$$

often written this way.

Important! This is for ~~European~~ options only.

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2): Multiple Underlyings

2 underlying assets R, S.

$$dS_t = \mu_1 S_t dt + \sigma_1 S_t dW_t.$$

$$dR_t = \mu_2 R_t dt + \sigma_2 R_t dB_t$$

We assume B and W are uncorrelated.

B, W BM's.

$V(S,R,t)$ is our European option

<u>EX</u> Magrabe option:

Owner has right, but not obligation to exchange a units of S for b units of R.

$$V(S,R,T) = \cancel{\max(aS_T - bR_T, 0)}$$
$$\max(bR_T - aS_T, 0).$$

Let derive PDE:

$$\pi = V - \Delta_S S - \Delta_R R.$$

2-D. Ito: $\cancel{dV = V_t dt + V_S}$

$$d\pi = V_t\, dt + V_S\, dS + V_R\, dR$$

$$+ \frac{1}{2}\sigma_1^2 S_t^2 \cancel{dt}\, V_{SS}\, dt + \frac{1}{2}\sigma_2^2 R_t^2 \cancel{dt}\, V_{RR}\, dt$$

$$- \Delta_S\, dS - \Delta_R\, dR.$$

$$= (V_S - \Delta_S)\, dS + (V_R - \Delta_R)\, dR.$$

$$+ \left(V_t + \frac{1}{2}\sigma_1^2 S_t^2 V_{SS} + \frac{1}{2}\sigma_2^2 R_t^2 V_{RR}\right) dt$$

set $\Delta_S = V_S$, $\Delta_R = V_R$.

$$d\pi = \left(V_t + \frac{1}{2}\sigma_1^2 S_t^2 V_{SS} + \frac{1}{2}\sigma_2^2 R_t^2 V_{RR}\right) dt$$

$\parallel$

$r\pi dt.$

$$= r\pi\, dt = r(V - V_S S - V_R R)\, dt$$

$$V_t + \frac{1}{2}\sigma_1^2 S_t^2 V_{SS} + \frac{1}{2}\sigma_2^2 R_t^2 V_{RR} + rV_S S + rV_R R$$
$$- rV = 0$$

Similar to 1-D BS

Important: If we assumed correlation:

$d[B, W] = \rho\, dt$ we would get an extra

↳ covariation.

term in the PDE:

$$\rho_t \sigma_1 \sigma_2 RS V_{RS}$$

<u>Ex</u> For the Magrabe option the sol'n

is.

$\cancel{a S N(d_1) - b R N(d_2)}$.

$b R N(d_1) - a S N(d_2)$.

$$d_1 = \frac{\ln\left(\frac{R}{S}\right) + (r + \frac{1}{2}\sigma_*^2)(T-t)}{\sigma_* \sqrt{T-t}}, \quad d_2 = d_1 - \sigma_* \sqrt{T-t}$$

$$\sigma_* = \sigma_1^2 - \rho\sigma_1\sigma_2 + \sigma_2^2$$

"aggregate volatility"