Last time: Black - Sholes - Merton.

~~$c(x$ $c(t,x) \longrightarrow$ Price of E~~

Consider Eur. call, strike K, mat T.

Asset $\longrightarrow$ Stock price is $GBM(\alpha, \sigma)$

$$dS = \alpha S\, dt + \sigma S\, dW$$

M.M. $\to$ Interest rate r. (borrow & sell).

① If $c(t,x)$ = Price of this call $\begin{pmatrix} \text{spot price} = x \\ \text{time} = t \end{pmatrix}$

$$\Rightarrow \partial_t c + r x \partial_x c + \frac{\sigma^2}{2} x^2 \partial_x^2 c = rc$$

$$\left(c(t,0) = 0, \quad c(T,x) = (x-K)^+, \ \cdots \right).$$

(2) If c solves the BSM PDE

$\Rightarrow c(t, S(t)) =$ AF Price of the E. call

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(1) Idea: (a) Itô formula

(b) Uniqueness of the Itô decomposition.

Check $\uparrow$:

Say $X = X(0) + B_1 + M_1$

& $X = X(0) + B_2 + M_2$.

Claim: $B_1 = B_2$ & $M_1 = M_2$.

$B_1, B_2 \longrightarrow$ finite 1^{st} variation

$M_1, M_2 \longrightarrow$ Mg.

Quick Pf:

$\Rightarrow B_1 + M_1 = B_2 + M_2.$

$\Rightarrow \underbrace{B_1 - B_2}_{B} = \underbrace{M_2 - M_1}_{M}.$

B ← finite 1st var

M ← M_q.

Know $B(0) = M(0) = 0$

(NTS $B(t) = M(t) = 0$ $t \geq 0$).

Know $EM(t)^2 = E[M,M]$

Since $M = B$, & $[B,B] = 0$

$\Rightarrow [M,M] = 0 \Rightarrow EM(t)^2 = 0$

$\Rightarrow M(t) = 0.$ //

Can solve B.S PDE & get a formula for c.

$$c(t,x) = x N(d_+) - K e^{-r(T-t)} N(d_-).$$

$$d_\pm = d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right)$$

$$\tau = T - t$$

Q: Price of a Euro. Put?

Put call parity: Portfolio = $+1$ call, -1 put.

$p(t, x)$ = price of the put (spot price x).
(same param as call).

$$p(T, x) = (K - x)^+$$

$$\Rightarrow c(T, x) - p(T, x) = (x - K)^+ - (K - x)^+$$

$$= x - K \quad \text{(forward contract).}$$

$$c(t, x) - p(t, x) = x - K e^{-r(T-t)}$$

$$\Rightarrow p(t, x) = c(t, x) + K e^{-r(T-t)} - x$$

GREEKS: (derivatives of c).

① Delta: $\partial_x c(t,x)$.

$\longrightarrow$ # of shares of the asset that should be in the R. Portfolio at time t is exactly $\boxed{\Delta(t) = \partial_x c(t, S(t))}$

(Delta Hedging rule)

($N' =$ derivative of N).

Formula: $\partial_x c = N(d_+) + \underbrace{x N'(d_+)\partial_x d_+ - K e^{-r\tau} N'(d_-)\partial_x d_-}_{\text{turns out } = 0}$

Get $\boxed{\partial_x c = N(d_+)}$.

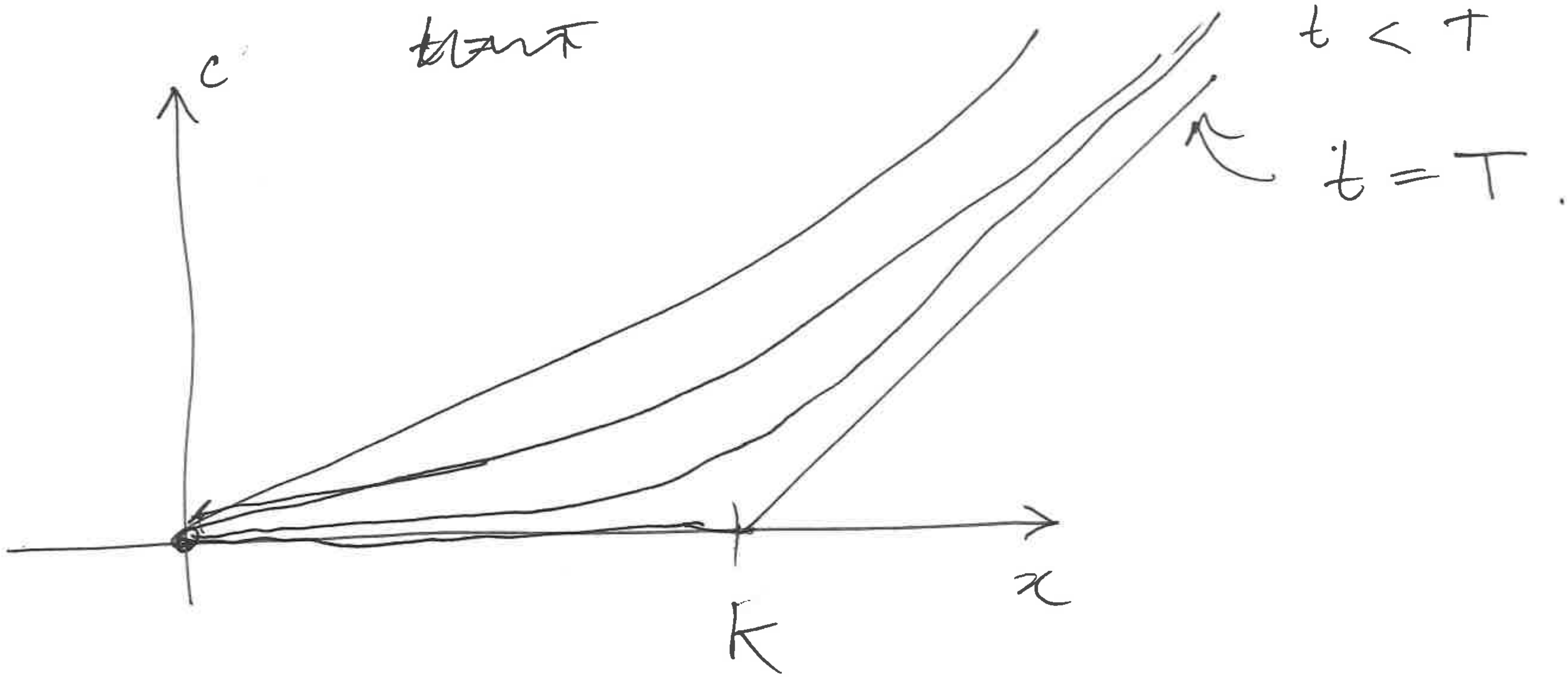
② **Gamma:** Second derivative of c wrt x. $(\partial_x^2 c)$.

Can check: $$\partial_x^2 c = N'(d_+)\, d_+' = \frac{1}{x \sigma \sqrt{2\pi\tau}} \exp\left(\frac{-d_+^2}{2}\right).$$

③ **Theta:** Time derivative of c.

$$\partial_t c = -rKe^{-r\tau} N(d_-) - \frac{\sigma x}{2\sqrt{\tau}} N'(d_+)$$

Proposition:
1. $c(t,x)$ is increasing as a fn of x $(\partial_x c > 0)$
2. $c(t,x)$ is decreasing as a fn of t. $(\partial_t c < 0)$
3. $c(t,x)$ is convex as a fn of x. $(\partial_x^2 c > 0)$.



Hedging a short call:

Sell E. call. $c(t,x)$.

Want to hedge:

R. Portfolio $\begin{cases} \to x\partial_x c & \text{in the asset} \\ \to & \text{Rest cash.} \end{cases}$

Cash value (of R. Portfolio).

$$= c(t,x) - x\,\partial_x c(t,x)$$

$$= \cancel{x N(d_+)} - K e^{-r\tau} N(d_-) - \cancel{x N(d_+)}$$

$$= -K e^{-r\tau} N(d_-) < 0.$$

Delta Neutral / long Gamma:

Say at time t, spot price x_0.

Short $\partial_x c(t,x_0)$ shares of stock

& buy 1 call option $(c(t,x_0))$.

Balance $M = x_0 \partial_x c(t, x_0) - c(t, x_0)$. (in M.M.)

Say spot price changes to x

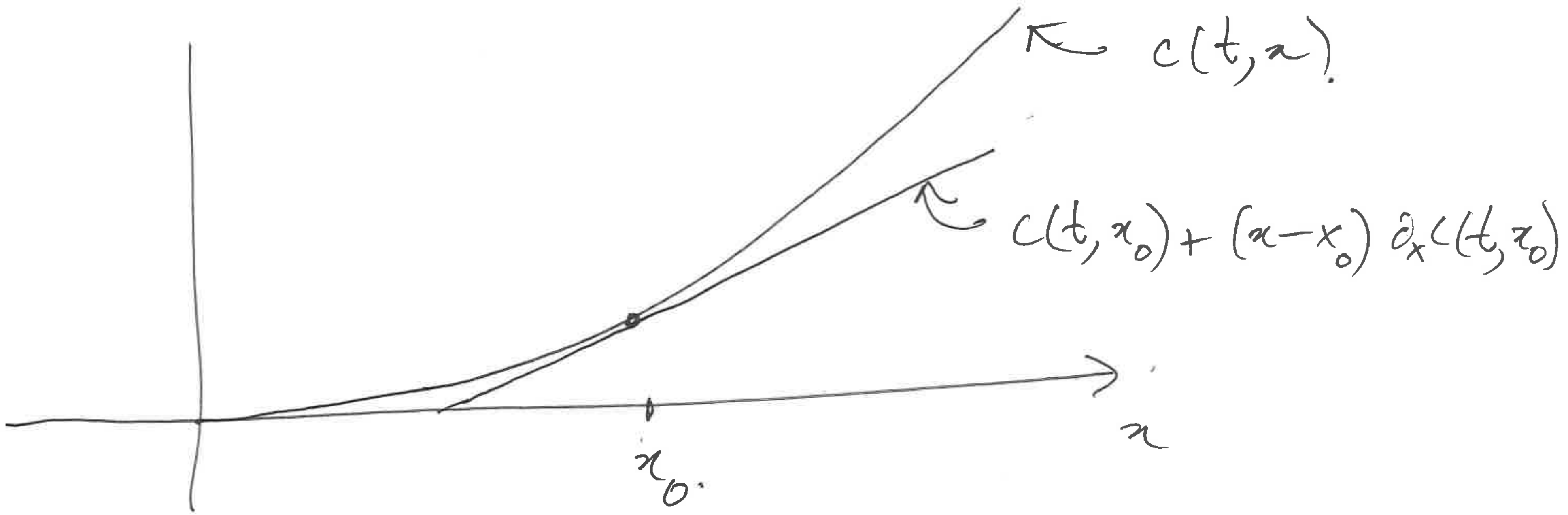
(& we hold our position of $\partial_x c(t, x_0)$ shares.)

New value of Pf:

$$c(t,x) - \partial_x c(t,x_0)\, x + M$$

$$= c(t,x) - x\, \partial_x c(t,x_0) + x_0 \partial_x c(t,x_0) - c(t,x_0)$$

$$= c(t,x) - \Big[\underbrace{c(t,x_0) + (x - x_0)\, \partial_x c(t,x_0)}_{\text{tangent line.}}\Big],$$



<u>Next Goal:</u> "Risk Neutral Pricing Formula".

Security $\to$ Payoff $V(T)$.

Price at time $t < T$ is $\tilde{E}\left(e^{-r(T-t)} V(S_T) \,\middle|\, \mathcal{F}_t\right)$

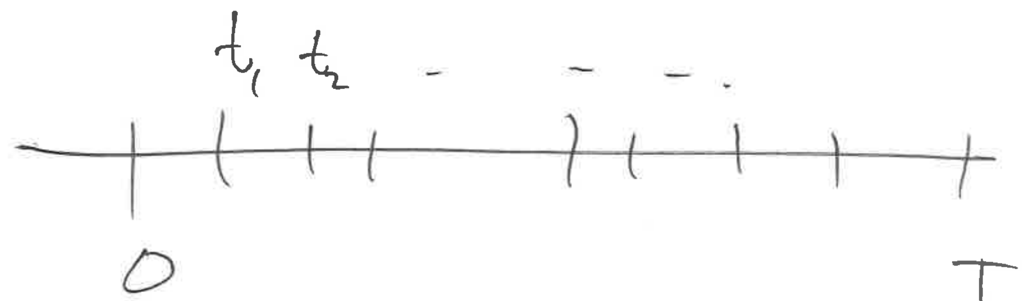
"conditional exp wrt "Risk Neutral Measure""

① Multi Dim. Itô formula. (today).

② Girsanov Theorem.

Joint Quadratic Variation.

X, Y two Itô processes.



$$\Delta_i X = X(t_{i+1}) - X(t_i).$$

$$\Delta_i Y = Y(t_{i+1}) - Y(t_i).$$

$$P = \{0 \cancel{\le} = t_0 < t_1 \cdots < t_n = T\}$$

$$\|P\| = \max_i \; t_{i+1} - t_i.$$

Def: The Joint Quadratic Variation between X & Y.

is defined by

$$[X, Y](T) = \lim_{\|P\| \to 0} \sum_i (\Delta_i X)(\Delta_i Y).$$

Trick: $4ab = (a+b)^2 - (a-b)^2$

$$\Rightarrow (\Delta_i X)(\cancel{X}\Delta_i Y) = \frac{1}{4}\left((\Delta_i (X+Y))^2 - (\Delta_i (X-Y))^2\right).$$

$$\Rightarrow 4[X, Y] = [X+Y, X+Y] - [X-Y, X-Y].$$

Proposition (Product Rule).

$$d(XY) = X\,dY + Y\,dX + d[X,Y].$$

Pf: $d(X+Y)^2 \overset{\text{Itô}}{=} 2(X+Y)\,d(X+Y) + d[X+Y, X+Y]$.

& $d(X-Y)^2 \overset{\text{Itô}}{=} 2(X-Y)\,d(X-Y) + d[X-Y, X-Y]$.

$$\Rightarrow 4\,d(XY) = d(X+Y)^2 - d(X-Y)^2.$$

$$= 4X\,dY + 4Y\,dX + 4\,d[X,Y].$$

//

Prop: Say X some Itô process. $\left.\begin{array}{l} \\ B \text{ has finite first var} \end{array}\right\}$ $[X, B] = 0$

Pf: $4[X,B] = \underbrace{[X+B, X+B]}_{} - \underbrace{[X-B, \cancel{B}X-B]}_{}.$

$= [X,X] - [X,X] = 0$

Thm (Multi Dim Itô formula).

$X_1, \dots X_n \to n$ Itô processes.

Let $X = (X_1, X_2, \dots X_n)$ (n-dim Ito process), Let f be a fn of t & $x_1, x_2 \cdots x_n$.

(f non - random).

① $\partial_t f$ is cts.

& ② $\partial_i f = \dfrac{\partial f}{\partial x_i}$ are all cts. $(i \in \{1, \dots n\})$.

③ $\partial_i \partial_j f = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$ are all cts.

$$d\Big(f(t, X(t))\Big) = \partial_t f(t, X(t))\, dt + \sum_{i=1}^{n} \partial_i f(t, X(t))\, dX_i(t) + \frac{1}{2}\left(\sum_{i=1}^{n} \sum_{j=1}^{n} \partial_i \partial_j f(t, X(t))\, d[X_i, X_j](t)\right.$$