BLACK SCHOLES. (MERTON)

Geometric . BM.

$S(t) \rightarrow$ price of a Stock at time t.

Constant Return rate: $dS(t) = \alpha S(t)\, dt$

(no noise).

Risky assets (Stocks)

$$dS(t) = \alpha S(t)\, dt + \sigma S(t)\, dW$$

$$S(t) = S(0) + \int_0^t \alpha S(s)\, ds + \int_0^t \sigma S(s)\, dW(s)$$

$\alpha \longrightarrow$ mean return rate.

$\sigma \longrightarrow$ volatility. (percent volatility).

S is a Geometric BM. $\quad S = GBM(\alpha, \sigma)$.

Let $Y = \ln(S)$

Itô:
$$dY = 0\,dt + \frac{1}{S}\,dS + \frac{-1}{2S^2}\,d[S,S].$$
$$= \alpha\,dt + \sigma\,dW(t) - \frac{1}{2}\sigma^2 \cancel{d(t,t)}\,dt$$
$$= \left(\alpha - \frac{\sigma^2}{2}\right)dt + \sigma\,dW.$$

$$\Rightarrow Y(t) = Y(0) + \cancel{\int_0^t} \left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W(t).$$

$$\boxed{\Rightarrow S(t) = \exp(Y(t)) = S(0) \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right) t + \sigma W(t)\right).}$$

Assume $S(t) \longrightarrow$ price of a Stock.

Consider European call strike K maturity T.

Theorem: Say we have an arbitrage free market.

with
1. Money Market $\rightarrow$ return rate r.
2. Stock $\rightarrow$ $S(t)$ $(GBM(\alpha, \sigma))$.

Consider a European call, strike K, mat T.

1. If $c = c(t, x)$ (non-random fn) is such that at any time $t \leq T$,

$$c(t, S(t)) = \text{AFP of the call.}$$

then

c satisfies.

(a) $\partial_t c + r x \partial_x c + \frac{\sigma^2 x^2}{2} \partial_x^2 c = r c$

(b) $c(t, 0) = 0$

(c) $c(T, x) = (x - K)^+$

B.S.M
Partial diff'tial equation.

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and (2) Converse: If c satisfies (a) $\rightarrow$ (c) above, then at any time t, $c(t, S(t)) = $ AFP of the call.

Assumptions: (1) Frictionless (no transaction costs).

(2) Liquidity (buy & sell fractional quantities of S).

(3) Borrowing & lending rate $= r$.

Remark: Boundary condition at $x = \infty$.

BSM PDE is (a) (b) (c) & (d)

(d) $$\lim_{x \to \infty} \cancel{\left(c(t,x) - K e^{-r(T-t)}\right) = 0}$$

$$\lim_{x \to \infty} \left[c(t,x) - \left(x - K e^{-r(T-t)} \right) \right] = 0$$

Reason: $\quad x \gg K$. (Deep in the money).

End

Guess: R- Portfolio should long the stock.
& short $Ke^{-r(T-t)}$ cash.

$$x - Ke^{-r(T-t)} = \text{value of } R \text{ portfolio} \quad (x \gg K).$$

$$\Rightarrow c(t,x) \approx x - Ke^{-r(T-t)} \quad (x \gg K).$$

Pf of thm (i). Assume c = AFP.

NTS c satisfies (a)

Let $X(t)$ = value of R. Portfolio at time t.

$$\text{R. Pf} = \begin{cases} \Delta(t) \text{ shares of the stock.} \\ \text{Rest cash. } (\#\text{ cash} = X(t) - \Delta(t) S(t)). \end{cases}$$

$$\Rightarrow dX = \Delta(t)\, dS(t) + r\big(X(t) - \Delta(t) S(t)\big)\, dt.$$

$$=$$

Assumption: $c(t, S(t)) = \text{AFP}$

$X(t) = \text{value of R-Portfolio}$

$\Longrightarrow c(t, S(t)) = X(t).$

$\Longrightarrow dX(t) = d\left(c(t, S(t))\right)$

Itô

(i) $dX(t) = \Delta(t)\, dS(t) + r\left(X(t) - \Delta(t) S(t)\right) dt.$

$$= \Delta(t)\left(\alpha S(t) dt + \sigma S(t) dW(t)\right) + r\left(\quad\right) dt$$

$$= \sigma \Delta(t) S(t)\, dW(t) + \left[(\alpha - r)\Delta(t) S(t) + rX\right] dt.$$

(2) $$d\big(c(t,S(t))\big) = \partial_t c(t,S(t))\,dt + \partial_x c(t,S(t))\,dS + \frac{1}{2}\partial_x^2 c(t,S(t))\,d[S,S]$$

$$= \partial_t c\,dt + \partial_x c\,(\alpha S\,dt + \sigma S\,dW) + \frac{1}{2}\partial_x^2 c\,\sigma^2 S^2\,dt$$

$$= \left(\partial_t c + \alpha\,\partial_x c\,S + \frac{1}{2}\partial_x^2 c\,\sigma^2 S^2\right)dt + \partial_x c\,\sigma S\,dW.$$

Since the Itô decomposition is unique.

we can equate the mg terms & the B.V. terms.

$$\Longrightarrow \cancel{r}\,\Delta(t)\,\cancel{S(t)} = \cancel{r}\,\cancel{S(t)}\;\underline{\partial_x c(t, S(t))}$$

$$\boxed{\Rightarrow \Delta(t) = \partial_x c(t, S(t))}$$

← Delta Hedging Rule.

Equate the dt terms:

$$\partial_t c + \alpha\, \partial_x c\, S + \frac{1}{2} \partial_x^2 c\, \sigma^2 S^2 =$$

$$= (\alpha - r)\underbrace{\Delta(t)}_{\partial_x c} S(t) + r \underbrace{X}_{c}$$

$$\Rightarrow \partial_t c + 0 + \frac{1}{2} \partial_x^2 c\, \sigma^2 S^2 = - r\, \partial_x c\, S + r c.$$

Write x instead of S.

$$\boxed{\Rightarrow \partial_t c + x r \partial_x c + \frac{1}{2}\sigma^2 x^2 \partial_x^2 c = r c.}$$

BSM PDE.

<u>NOTE:</u> $c(t,x)$ depends on K, σ, T, x, t, r

but $\boxed{\text{not on } \alpha.}$

<u>Converse:</u> Suppose c satisfies BSM PDE

Need to show $c(t, S(t)) =$ AFP of the call.

Let $X(t)$ be the ~~rep~~ wealth of the R. Portfolio.

↳ $X(T) = (S(T) - K)^+$.

NTS: $X(t) = c(t, S(t))$.

Choose $\Delta(t) = \partial_x c(t, S(t))$.

I.E. Chosing X to be a portfolio that holds

$\Delta(t)$ shares of S $\left(\Delta(t) = \partial_x c(t, S(t))\right)$.

& $X(0, \cancel{\ }) = c(0, S(0))$.

Will show X is a replicating portfolio & $X(t) = c(t, S(t))$ for all t.

(Clever trick → in notes.

compute $d\left(e^{-rt}X(t)\right)$).

Compute $dX = \Delta(t)\,dS + r\left(X - \Delta(t)S(t)\right)dt$

& $d\left(c(t,S(t))\right) = \partial_t c\,dt + \partial_x c\,dS + \frac{1}{2}\partial_x^2 c\,d[S,S]$.

$$= \partial_t c\,dt + \partial_x c\left(\alpha S\,dt + \sigma S\,dW\right) + S^2\frac{\sigma^2}{2}\partial_x^2 c\,dt.$$

$$= \left(\partial_t c + \alpha S\,\partial_x c + \frac{\sigma^2 S^2}{2}\partial_x^2 c\right)dt + \sigma S\,\partial_x c\,dW.$$

$$\overset{\text{BS PDE}}{=} \left(rc + (\alpha - r)S\,\partial_x c\right)dt + \sigma S\,\partial_x c\,dW.$$

$$dX = \Delta(t)\left(\alpha S\,dt + \sigma S\,dW\right) + r\left(X - \Delta(t)S(t)\right)dt$$

$$= \left(\cancel{\alpha\Delta S +} \; (\alpha - r)\underbrace{\Delta(t)}_{\partial_x c} S(t) + rX\right)dt + \Delta(t)\sigma S(t)\,dW.$$

Set $Y(t) = X(t) - c(t, S(t))$.

$$\Rightarrow dY = 0\,dW + (rX - rc)\,dt$$

$$= r(X - c)\,dt = rY\,dt.$$

$$\Rightarrow \cancel{Y(t) = Y(T)e^{r(t-T)}} \quad Y(t) = \underline{\underline{Y(0)}}\,e^{rt} = 0$$

$\Rightarrow Y(t) = 0$ for all t

$$\boxed{\Rightarrow X(t) = c(t, S(t)).}$$

$\Rightarrow X(T) = \underbrace{c(\cancel{t}T, S(T))} = (S(T) - K)^+$

$\Rightarrow X$ is a replicating portfolio.

$\Rightarrow \square \Rightarrow c(t, s(t))$ is the ~~AFP~~ AFP!!.