Q: $r \leq s < t$.

$W \longrightarrow$ std BM.

Compute $E\big(W(r)\,W(s)\,W(t)\big)$. ($\leftarrow$ ~~Guess: r~~ Because ~~$E\big(W(s)\,W(t)\big) = s \wedge t$~~).

Sol: $E\big(W(r)\,W(s)\,W(t)\big)$

$$= E\Big(E\big(W(r)\,W(s)\,W(t) \mid \mathcal{F}_s\big)\Big)$$

$$= \cancel{E\,W(r)\,E\big(W(s)\,W(t) \mid \mathcal{F}_{r\wedge s}\big)}$$

$$= E\Big(W(r)\,W(s)\;E\big(W(t) \mid \mathcal{F}_s\big)\Big)$$

$$= E\big(W(r)\,W(s)^2\big).$$

$$= \mathbb{E}\ E\left(W(r)\,W(s)^2 \,\middle|\, \mathcal{F}_r\right).$$

$$= E\,W(r)\ E\left(W(s)^2 \,\middle|\, \mathcal{F}_r\right)$$

$$= E\,W(r)\left[E\left(\underbrace{W(s)^2 - s}_{r} + s \,\middle|\, \mathcal{F}_r\right)\right].$$

$$= E\,W(r)\left[\,W(r)^2 - r + s\,\right].$$

$$= 0$$

Q2: $X(t) = \int_0^{W(t)} e^{-s^2}\, ds.$ $\quad Z(t) = \dfrac{t}{X(t)}.$

$$Y(t) = \exp\left(\int_0^t W(s)\, ds \right).$$

Goal: Find Itô decomposition & Q.V.

→ Write as a mg + a process of finite 1st var.

$\underbrace{\hspace{3cm}}$

Ito Int (mg)

+ Riemann Int (BV).

① X : $f(t,x) = \int_0^x e^{-s^2}\,ds$.

$X(t) = f(t, W(t))$ & apply Ito.

$$\partial_t f = 0 \qquad \partial_x f = e^{-x^2}$$

$$\partial_x^2 f = e^{-x^2}(-2x).$$

$$\Rightarrow dX = \partial_t f\,dt + \partial_x f\,dW + \frac{1}{2}\partial_x^2 f\,d[W,W].$$

$$= 0 + e^{-W(t)^2}\,dW(t) - \frac{1}{\cancel{2}}\cancel{2}\,W(t)\,e^{-W(t)^2}\,dt$$

$$\Rightarrow X(t) = \underbrace{-\int_0^t W(s)\, e^{-W(s)^2}\, ds}_{\substack{BV \\ (\text{finite first var})}} + \underbrace{\int_0^t e^{-W(s)^2}\, dW(s)}_{Mg}.$$

$$[X,X](t) = \int_0^t e^{-2W(s)^2}\, ds$$

~~X(t) = If~~

If instead $X(t) = \int_0^{W(t)} e^{-W(\cancel{t}s)^2}\, dW(s)$

Q: Ito decomp:

Try $f(t,x) = \int_0^{x} e^{-W(s)^2}\,dW(s)$.

Use Itô to figure out $d\,f(W(t))$.

Won't Work: (1) For Ito need $\partial_t f$, $\partial_x f$ & $\partial_x^2 f$ exist (& be cts).

Not true here!

f is not diff (even once) wrt x.

$\boxed{\text{(2) Also } f \text{ is random!}}$ ← Fixable.

$$Y(t) = \exp\left(\int_0^t W(s)\,ds\right).$$

Q: Itô decomposition? & Q.V.

$$\cancel{f(t,x) = \exp\left(\int_0^t W(s)\,ds\right)}$$

$$\boxed{\begin{array}{l} \text{Ito twice:} \quad X(t) = \int_0^t W(s)\,ds. \\ \qquad f(t,x) = e^x \qquad Y(t) = f(t, X(t)) \\ \text{Apply Ito: Guaranteed to work.} \end{array}}$$

Y is diff as a fn of t $\Rightarrow$ Y has finite first variation.

Ito decomposition: $Y(t) = \underbrace{Y(t)}_{\text{BV part}} + \underbrace{0}_{\text{Mg part}} \Rightarrow [Y, Y] = 0$

If B is diff as a fn of time.

$$B(t) = B(0) + \underbrace{\int_0^t (\partial_t B)\, ds}_{\text{Finite first var.}}$$

$$\left(\text{Need: } \int_0^t |\partial_t B|\, ds < \infty \quad \text{almost surely} \right).$$

Q3: $X(t) = \int_0^t W(s)\, ds$ & $Y(t) = \int_0^t W(s)\, dW(s)$.

Given $s \leq t$, compute $E(X(t) \mid \mathcal{F}_s)$ & $E(Y(t) \mid \mathcal{F}_s)$.

Note $\quad E\left(Y(t) \mid \mathcal{F}_s\right) = Y(s)$

(Y is an Ito int $\Rightarrow$ Y is a mg).

$$E\left(X(t) \mid \mathcal{F}_s\right) = E\left(\int_0^t W(r)\,dr \,\Big|\, \mathcal{F}_s\right).$$

$$= \int_0^t E\left(W(r) \mid \mathcal{F}_s\right) \, dr$$

$$= \int_0^s (\;) + \int_s^t (\;)$$

$$= \int_0^s W(r)\,dr + \int_s^t W(s)\,dr = \int_0^s W(r)\,dr + W(s)(t-s)$$

Q5: $\sigma = \sigma(t)$ $\boxed{\text{non-random}}$.

$$X(t) = \int_0^t \sigma(s)\, dW(s)$$

Q: $\lambda,\ s,t \geq 0,\quad 0 \leq s < t.$ ~~Compute~~

Compute $E\left(e^{\lambda(X(t)-X(s))} \,\middle|\, \mathcal{F}_s \right).$

Ito: $e^{\lambda X(t)}$

$$d\left(e^{\lambda X(t)}\right) = \lambda e^{\lambda X(t)} dX + \frac{\lambda^2}{2} e^{\lambda X(t)} d[X,X].$$

$$= \lambda e^{\lambda X(t)} \sigma(t)\, dW(t) + \frac{\lambda^2}{2} e^{\lambda X(t)} \sigma(t)^2\, dt$$

$$\Rightarrow e^{\lambda X(t)} - e^{\lambda X(s)} = \int_s^t \lambda e^{\lambda X(r)} \sigma(r)\, dW(r) + \frac{\lambda^2}{2} \int_s^t e^{\lambda X(r)} \sigma(r)^2 \, dr .$$

$$\Rightarrow \cancel{E}\, e^{\lambda(X(t) - X(s)} - 1 = e^{-\lambda X(s)} \int_s^t (\ )\, dW(r) + \frac{\lambda^2}{2} \int_s^t e^{\lambda(X(r) - X(s))} \sigma(r)^2 \, dr .$$

$$\Rightarrow E\left(e^{\lambda(X(t)-X(s))} \,\middle|\, \mathcal{F}_s\right) = 1 + 0$$

$$+ \frac{\lambda^2}{2} \int_s^t E\left(e^{\lambda(X(r)-X(s))} \,\middle|\, \mathcal{F}_s\right) \sigma(r)^2 \, dr \qquad (*)$$

Let $\varphi(t) = E\left(e^{\lambda(X(t)-X(s))} \,\middle|\, \mathcal{F}_s\right)$.

Then. $(*) \Rightarrow \varphi(t) = 1 + \frac{\lambda^2}{2} \int_s^t \sigma(r)^2 \varphi(r) \, dr$.

$$\Rightarrow \partial_t \varphi(t) = \frac{\lambda^2}{2} \sigma(t)^2 \varphi(t)$$

$$\Rightarrow \varphi(t) = \exp\left(\frac{\lambda^2}{2} \int_s^t \sigma(r)^2 \, dr \right).$$

$$\boxed{\Rightarrow E\left(e^{\lambda(X(t)-X(s))} \,\middle|\, \mathcal{F}_s\right) = \exp\left(\frac{\lambda^2}{2}\int_s^t \sigma(r)^2\,dr\right).}$$

Q: If $r \leq s$: Compute.

$$E\left(e^{\lambda X(r) + \mu(X(t)-X(s))}\right) =$$

$$= E\,E\left(e^{\lambda X(r) + \mu(X(t)-X(s))} \,\middle|\, \mathcal{F}_s\right).$$

$$= E\,e^{\lambda X(r)}\, E\left(e^{\mu(X(t)-X(s))} \,\middle|\, \mathcal{F}_s\right)$$

$$= E\,e^{\lambda X(r)} \exp\left(\frac{\mu^2}{2}\int_s^t \sigma(u)^2\,du\right).$$

$$= \exp\left(\frac{\lambda^2}{2}\int_0^r \sigma(u)^2\,du\right)\cdot \exp\left(\frac{\mu^2}{2}\int_s^t \sigma(u)^2\,du\right).$$

Q: Joint dist of $(X(r), X(t)-X(s))$.

Jointly Normal: Mean 0

Covariance
$$\begin{pmatrix} \int_0^r \sigma(u)^2\,du & 0 \\ 0 & \int_s^t \sigma(u)^2\,du \end{pmatrix}.$$

(d) Suppose $\sigma(x) = \pm 1$.

$\left(\Rightarrow d[X,X] = dt \text{ \& } X \text{ is a mg} \right)$.

Levy's Criterion: X is a std BM.

Pf: Know $X(t) - X(s) \sim N(0, t-s)$.

Also, for $r \le s$, $X(t) - X(s)$ is ind of $X(r)$.

$\Rightarrow X(t) - X(s)$ is ind of $\mathcal{F}_s$.

$\Rightarrow$ Ind inc $\Rightarrow$ done!

Q:

$$X(t) = \int_0^t s \, dW(s)$$

$$Y(t) = \int_0^t W(s) \, ds.$$

Compute $E X(t)^n$ & $E Y(t)^n$.

(a) Compute $E X(t)^n$:

Compute $E X(t)^2 \overset{\text{(Ito Isom)}}{=} E\left(\int_0^t s^2 \, ds\right) = \frac{t^3}{3}.$

Know $\int_0^t s \, dW(s)$ is normally dist (prev problem).

with mean 0 & variance $\frac{t^3}{3}$.

Know $X(t) \sim N\left(0, \frac{t^3}{3}\right)$.

Compute MGF $\Rightarrow$ Find EX^n.

Compute $E\,Y(t)^n$ $\quad \left(Y(t) = \int_0^t W(s)\, ds \right)$.

Note $$Y(t) = \lim_{\|P\| \to 0} \left.\underbrace{\sum \underbrace{W(t_i)(t_{i+1} - t_i)}_{\text{Normal.}}}_{\text{Normal.}}\right\} \text{Normal.}$$

Expect $Y(t)$ is normal.

$$EY(t) = \int_0^t EW(s)\,ds = 0$$

Compute $EY(t)^2$: (Once we know $EY(t)^2$.
Use MGF of normal to find $EY(t)^n$).

$\to$ $EY(t)^2$: Method 1:

$$EY(t)^2 = E\left(\int_0^t W(s)\,ds\right)^2 = E\left(\int_0^t W(s)\,ds\right)\left(\int_0^t W(s)\,ds\right).$$

$$= E\left(\int_0^t W(s)\,ds\right)\left(\int_0^t W(r)\,dr\right).$$

$$= E\int_0^t\int_0^t W(s)\,W(r)\;ds\,dr$$

$$= \int_0^t\int_0^t \underbrace{E\,W(s)\,W(r)}_{s\wedge r}\;ds\,dr.$$

& evaluate this integral.

Trick ②: Compute $E\,Y(t)^2$.

$$Y(t) = \int_0^t W(s)\,ds \;\Rightarrow$$

guess

$$d(\quad) = \underbrace{W(t)\,dt} + (\quad)\,dW.$$

$$d(t\,W(t)) = W(t)\,dt + t\,dW + 0$$

$$\Rightarrow t\,W(t) = \int_0^t \cancel{W(t)\,dt}\;W(s)\,ds + \int_0^t s\,dW(s).$$

$$\Rightarrow \boxed{Y(t) = \int_0^t s\, dW(s) - t\, W(t)}.$$

$$Y(t)^2 = \cancel{t}\left(\int_0^t s\, dW(s)\right)^2 + t^2 W(t)^2 - 2\underbrace{t W(t) \int_0^t s\, dW(s)}_{}.$$

$$E\, Y(t)^2 = E$$

$$d\, Y^2 = 2 Y\, dY + \underbrace{d[Y, Y]}_{0}.$$

$$Y(t)^2 = 2 \int_0^t Y\, dY.$$

Try Trick 2 again:

$$d\left(W(t)^3\right) = 0\,dt + 3W(t)^2 dW + 3\underbrace{W(t)\,dt}.$$

$$\Rightarrow 3Y(t) = W(t)^3 - 3\int_0^t W(t)^2 dW.$$

Compute $E\left(W(s)W(t) \mid \mathcal{F}_\tau\right)$ $\qquad \tau < s < t$.

↳ Guess: $\boxed{s \wedge t - \tau}$ ← Wrong.

Check: $E\left(W(s)W(t) \mid \mathcal{F}_\tau\right) = E\left(E\left(W(s)W(t) \mid \mathcal{F}_s\right) \mid \mathcal{F}_\tau\right)$

$$= E\left(W(s)\, E\left(W(t) \mid \mathcal{F}_s\right) \mid \mathcal{F}_\tau\right).$$

$$= E\left(W(s)^2 \mid \mathcal{F}_\tau\right).$$

$$= E\left(W(s)^2 - s + s \mid \mathcal{F}_\tau\right)$$

$$\boxed{= W(\tau)^2 - \tau + s}$$

Tower Property.

$$\mathcal{F}_1 \subseteq \mathcal{F}_2.$$

$$E(X \mid \mathcal{F}_1) = E\big(E(X \mid \mathcal{F}_2) \mid \mathcal{F}_1 \big).$$

Q: $Z(t) = \dfrac{t}{X(t)}$, $X(t) = \displaystyle\int_0^{W(t)} e^{-s^2}\, ds$.

Find Ito decomp of Z.

① Recall: $dX = e^{-W(t)^2}\, dW - W(t) e^{-W(t)^2}\, dt$

dZ:

$$dZ = \frac{1}{X(t)}\,dt + \left(\frac{-t}{X(t)^2}\right) dX + \frac{1}{2}\left(\frac{2t}{X(t)^3}\right) d[X,X].$$

(Pretend $X \neq 0$).

$$= \frac{dt}{X} - \frac{t}{X(t)^2}\left(e^{-W(t)^2} dW - W(t) e^{-W(t)^2} dt\right).$$

$$+ \frac{t}{X(t)^3}\, e^{-2W(t)^2} dt$$

$$= \left(\frac{1}{X} + \frac{tW(t)}{X(t)^2}\, e^{-W(t)^2} + \frac{t}{X(t)^3}\, e^{-2W(t)^2}\right) dt$$

$$\left(-\frac{t}{X(t)^2}\, e^{-W(t)^2}\right) dW$$

$\underline{Q\,2b]}$ $$X(t) = \left(W(t) + \frac{t^2}{2}\right) \cdot \exp\left(-\int_0^t s\,dW - \frac{t^2}{2}\right).$$

Q: Is X ~~is~~ a $\underline{mg}$.

Compute $E(X(t) \mid \mathcal{F}_s)$.

Compute $$E\left(W(t) \exp\left(-\int_0^t \cancel{s}\, dW(\cancel{s})\right) \,\middle|\, \mathcal{F}_s \right).$$

$$= E\left(\left(W(t) - W(s) + W(s)\right) \cdot \exp\left(-\int_0^s r\, dW(r)\right) \exp\left(-\int_s^t r\, dW(r)\right) \,\middle|\, \mathcal{F}_s \right)$$

$$= \left[E\left(\left(W(t)-W(s)\right) \cdot \exp\left(-\int_s^t r\,dW(r)\right)\right)\right] \exp\left(-\int_0^s r\,dW(r)\right).$$

$$+\, E\left(\exp\left(-\int_s^t r\,dW(r)\right)\right) W(s) \cdot \exp\left(-\int_0^s r\,dW(r)\right).$$

$$\vdots$$

4 a) Fix $T > 0$, f some cts fn.

$$E\left(f(W(T)) \mid \mathcal{F}_t\right) = \varphi(t, W(t)).$$

$\boxed{\text{Claim:}}$ $\partial_t \varphi + \frac{1}{2} \partial_x^2 \varphi = 0.$

Bad way to check: (1) Look up formula for φ.

(2) Let $X(t) = E\left(f(W(T)) \mid \mathcal{F}_t\right)$.

X is adapted.

$\boxed{\text{Claim } X \text{ is a mg!}}$

Check: $E(X(t) \mid \mathcal{F}_s) =$

$$E\Big(E(f(W(T)) \mid \mathcal{F}_t) \Big| \mathcal{F}_s \Big).$$

$$\stackrel{\text{Tower}}{=} E\big(f(W(T)) \mid \mathcal{F}_s\big) = X(s).$$

But: $X(t) = \varphi(t, W(t))$.

$$dX = \partial_t \varphi\, dt + \partial_x \varphi\, dW + \frac{1}{2} \partial_x^2 \varphi\, dt$$

$$= \left(\partial_t \varphi + \frac{1}{2} \partial_x^2 \varphi\right) dt + \cancel{dW}\, \partial_x \varphi\, dW.$$

X mg $\Rightarrow \partial_t \varphi + \frac{1}{2} \partial_x^2 \varphi = 0$!!

Uncorrellated $\not\Longrightarrow$ independent.

No.

If $M(t) - M(s)$ is is ind of $\mathcal{F}_s$

$$\Rightarrow E\, f(M(s))\, g(M(t)-M(s)) = (E M(s))\, E(M(t)-M(s)).$$

<u>Claim:</u> $M(t)-M(s)$ is NOT ind of $\mathcal{F}_s$.

$$E\, M(s)^2\, (M(t)-M(s))^2$$

$$\overset{\text{ind}}{\neq} (E M(s)^2)\, E\big((M(t)-M(s))^2\big).$$

1. $$X(t) = \int_0^{W(t)} W(s)\, dW(s).$$

(Can't Ito).

2. $$\int_0^{W(t)} W(s)\, ds.$$ $\left(f(x) = \int_0^x W(s)\, ds \right.$

<u>is NOT</u> C^2.

$$\partial_x f = \cancel{W(t)}\ W(x)$$

$X \longrightarrow$ adapted

$X(t)$ is $\mathcal{F}_t$ measurable.

$$\{X(t) < a\} \in \mathcal{F}_t.$$

Say $X(t)$ is not random.

$$\{X(t) < a\} = \begin{cases} \Omega & \in \mathcal{F}_t \\ \phi & \in \mathcal{F}_t. \end{cases}$$