Assignment due Tuesday

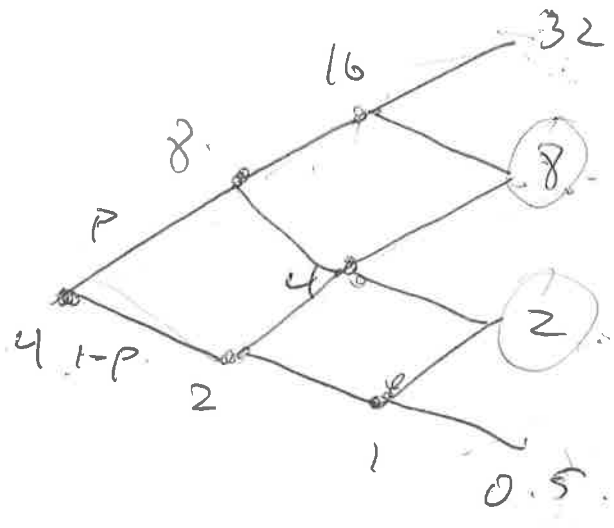
Midterm on Thursday.

<u>Extra OH.</u>

Monday:

NY Students 2:30 → 3:30 on Canvas

Pitt students 5:30 - 6:30



$C_1 = \{S_3 = 32\}$

$C_2 = \{S_3 = 8\}$

$C_3 = \{S_3 = 2\}$

$C_4 = \{S_3 = 0.5\}$

Find $\mathbb{E}[S_2 | S_3]$.

$$\mathbb{E}[S_2 | S_3] = \sum_{j=1}^{4} c_j \mathbb{1}_{C_j}.$$

To find c_i's you need to use PAC.

$$\mathbb{E}[S_2 \mathbb{1}_{C_1}] = \mathbb{E}[\mathbb{E}[S_2 | S_3] \mathbb{1}_{C_1}].$$

$$\mathbb{E}[S_2 \mathbb{1}_{C_2}] = \mathbb{E}[\mathbb{E}[S_2 | S_3]\, \mathbb{1}_{C_2}] = \mathbb{E}[C_2 \mathbb{1}_{C_2}] = C_2\, \mathbb{P}(C_2)$$

$$= C_2(p^2 q + p^2 q + p^2 q) = 3 C_2 p^2 q$$

$$\cancel{\mathbb{E}[S_2 \mathbb{1}_{C_2}] = 16 p^2 q + 4}$$

$$S_2 \mathbb{1}_{C_2} = \begin{cases} 16 & \text{if } HHT \\ 4 & \text{if } HTH \text{ or } THH \\ 0 & \text{o.w} \end{cases}$$

$$\mathbb{E}[S_2 \mathbb{1}_{C_2}] = 16 p^2 q + 8 p^2 q = 24 p^2 q$$

$$C_2 = 8$$

etc...

Today:

Ito Integration

Ito's formula.

Ito of Δ_t is $\int_0^t \Delta_s \, dW_s := \lim_{n\to\infty} \sum_{i=0}^{n-1} \Delta_{\frac{ti}{n}} \left(W_{\frac{t(i+1)}{n}} - W_{\frac{ti}{n}} \right)$

For this to make sense

Δ_t must be adapted.

and $\int_0^T \Delta_s^2 \, ds < \infty$.

X_t is an Ito process if.

$\sigma_t(W_t)$

$$X_t = X_0 + \underbrace{\int_0^t b_t\,dt}_{\text{drift}} + \underbrace{\int_0^t \sigma_t W_t}_{\text{volatility}} . \qquad (6).$$

Ito Isometry.

(W_t).

$$\mathbb{E}\left[\left(\int_0^T \sigma_t\,dW_t\right)^2\right] = \mathbb{E}\left[\int_0^T \sigma_t^2\,dt\right].$$

shorthand for (6).

$$dX_t = b_t\,dt + \sigma_t\,dW_t$$

X_t is an Ito process. If I apply some function to it, will it still be an Ito process?

And if so what is the decomposition?

<u>Ito's Lemma.</u> $f : \mathbb{R}^2 \to \mathbb{R}$. $\underline{f \in C^{1,2}}$. ← once continuously in time. ← twice " — " in 2nd variable.

$$df(t, X_t) = f_t\,dt + f_x\,dX_t + \tfrac{1}{2} f_{xx}\,d[X,X]_t.$$

$x,\ y.$ $\quad df(x,y) = f_x\,dx + f_y\,dy.$

$$\int_0^t \Delta_t\, dX_t = \int_0^t \Delta_s b_s\, ds + \int_0^t \Delta_s \sigma_s\, dW_s.$$

$$\text{if } X_t = \int_0^t b_t\, dt + \int_0^t \sigma_t\, dW_t.$$

Def'n/Facts.

$$[X,X]_t = \int_0^t \sigma_s^2\, ds \;\Big|\; d[X,X]_t = \sigma_t^2\, dt.$$

Ito's Formula becomes.

$$df(t, X_t) = \left(f_t + \frac{1}{2} f_{xx} \sigma_t^2\right) dt + f_x \, dX_t.$$

<u>Ex</u> Find $\int_0^t W_s \, dW_s$. Two different ways.

Naive guess: If we integrate $\int_0^t x \, dx = \frac{t^2}{2}$

$\hookrightarrow$ is $\frac{W_t^2}{2}$, but it's wrong.

(i) "Limit way". $\lim_{n \to \infty} \sum_{i=0}^{n-1} W_{\frac{ti}{n}} \left(W_{\frac{t(i+1)}{n}} - W_{\frac{ti}{n}} \right)$.

$$= \lim_{n \to \infty} \sum_{i=0}^{n-1} \left(\frac{1}{2} \left[W_{\frac{t(i+1)}{n}} + W_{\frac{ti}{n}} \right] - \frac{1}{2} \left[W_{\frac{t(i+1)}{n}} - W_{\frac{ti}{n}} \right] \right) \left(W_{\frac{t(i+1)}{n}} - W_{\frac{ti}{n}} \right)$$

$$= \lim_{n\to\infty}\left(\underbrace{\sum_{i=0}^{n-1} \frac{1}{2}\left(W_{\frac{t(i+1)}{n}}^2 - W_{\frac{ti}{n}}^2\right)}_{\text{II}\ \ \mathrm{I}} - \underbrace{\frac{1}{2}\sum_{i=0}^{n-1}\left(W_{\frac{t(i+1)}{n}} - W_{\frac{ti}{n}}\right)^2}_{\text{II}\ \ \mathrm{II}} \right).$$

$$\mathrm{I} = \cancel{\Sigma}\ \frac{1}{2}\left(W_t^2 - W_0^2\right) = \frac{1}{2}W_t^2$$

$$\mathrm{II} \quad W_{\frac{t(i+1)}{n}} - W_{\frac{ti}{n}} \sim N\left(0, \frac{t}{n}\right).$$

$$\text{for } i \neq j \quad W_{\frac{t(i+1)}{n}} - W_{\frac{ti}{n}} \perp\!\!\!\perp W_{\frac{t(j+1)}{n}} - W_{\frac{tj}{n}}.$$

$$\mathrm{II} = \frac{1}{2}\sum_{i=0}^{n-1}\frac{Z_i^2}{n}\, n \quad Z_i \sim N\left(0, \frac{t}{n}\right) \text{ IID}.$$

$$\cancel{\Sigma} \lim_{n\to\infty} \mathrm{II} = \frac{1}{2}\lim_{n\to\infty} \mathbb{E}[Z_1^2]\, n = \frac{t}{2}.$$

$$\therefore \int_0^t W_s \, dW_s = \frac{1}{2}(W_t^2 - t).$$

(ii) Ito's Lemma — use our initial guess $\frac{1}{2}W_t^2$.

i.e. use Ito's Lemma on $\frac{1}{2}W_t^2$.

$$f(t,x) = \frac{1}{2}x^2$$

$$f_t = 0, \quad f_x = x, \quad f_{xx} = 1$$

$$\frac{1}{2}W_t^2 = 0 + \int_0^t 0 \, dt + \int_0^t W_s \, dW_s + \frac{1}{2}\int_0^t dt$$

$$= \int_0^t W_s \, dW_s + \frac{t}{2}.$$

$$\Rightarrow \int_0^t W_s \, dW_s = \frac{1}{2}(W_t^2 - t).$$

Ex 2. $dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$ (**)

We want to find a process S_t that satisfies (**).

This is called a SDE, and this specific equation is called Geometric Brownian Motion (GBM).

Comments

i). Black-Scholes model uses this to model stock price fluctuation.

ii) μ is the return of the stock
σ^2 is the volatility.

Today we want to find S_t.

To get some intuition let's "turn off volatility"

pretend $\sigma = 0$.

$$dS_t = \mu S_t dt \quad \rightarrow \quad \frac{dS_t}{dt} = \mu S_t.$$

$$S_t = S_0 e^{\mu t}.$$

but a good ansatz in the general case would be.

$$S_0 e^{\mu t + \sigma W_t}.$$

Let's apply Ito's Lemma $f(t, x) = e^{\mu t + \sigma x}$.

$$f_t = \mu f, \quad f_x = \sigma f, \quad f_{xx} = \sigma^2 f.$$

$$d f(t, W_t) = \mu f dt + \sigma f dW_t + \frac{1}{2}\sigma^2 f dt$$

we wanted $f(t, W_t) = S_t$.

$$dS_t = \mu S dt + \sigma S dW_t + \frac{1}{2}\sigma^2 S dt.$$

Turns out solution is

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}.$$

Check at home that this solves $(*)$

Ex. $dS = \mu S dt + \sigma S dW_t$. $\quad \underset{\overset{\|}{f(t,x)}}{f(x)} = x^p$, ~~$p$~~.

find $d(S^p)$.

$$d(S^p) = 0 dt + p S_t^{p-1} dS_t + \frac{p(p-1)}{2} S_t^{p-2} d[S, S]_t$$

$$S_t = S_0 + \int_0^t \mu S_u du + \int_0^t \sigma S_u dW_u.$$

$$[S, S]_t = \int_0^t \sigma^2 S_u^2 du \Rightarrow d[S, S]_t = \sigma^2 S_t^2 dt.$$

$$= p S_t^{p-1}\left(\mu S_t\, dt + \sigma S_t\, dW_t\right) + \frac{p(p-1)}{2} S_t^{p-2} (\sigma S_t)^2\, dt.$$

$$= S_t^p \left(\left(p\mu + \frac{p(p-1)}{2}\sigma^2 \right) dt + p\sigma\, dW_t \right).$$

S^p

for what p is this a martingale?

martingale $\Longleftrightarrow$ "dt term is 0".

need $p\mu + \frac{p(p-1)}{2}\sigma^2 = 0$.

$\Rightarrow p = 0$ or $p = 1 - \frac{2\mu}{\sigma^2}$

Ex. Optional problem 3a) and 3b).

$$M_t = \int_0^t W_u \, dW_u \qquad s < t \qquad \text{find } \mathbb{E}[(M_t - M_s)^2 \mid \mathcal{F}_s].$$

$dM_t = W_t \, dW_t.$

For any martingale M:

$$\mathbb{E}[(M_t - M_s)^2 \mid \mathcal{F}_s] = \mathbb{E}[M_t^2 \mid \mathcal{F}_s] - 2\,\mathbb{E}[M_t M_s \mid \mathcal{F}_s] + \mathbb{E}[M_s^2 \mid \mathcal{F}_s].$$

$$= \mathbb{E}[M_t^2 \mid \mathcal{F}_s] - 2M_s^2 + M_s^2$$

$$= \mathbb{E}[M_t^2 - M_s^2 \mid \mathcal{F}_s].$$

$\mathbb{E}[M_t^2 \mid \mathcal{F}_s]$

One way is to subtract QV of M_t.

because we know $M_t^2 - [M,M]_t$ is a martingale.

$$[M,M]_t = \int_0^t W_u^2 \, du.$$

Other method is Itô's Lemma:

$$f(t,x) = x^2.$$

$$df(t,M_t) = 2M_t \, dM_t + 2\left(\tfrac{1}{2}\right) d[M,M]_t$$

$$= 2M_t W_t \, dW_t + W_t^2 \, dt.$$

$$M_t^2 = \int_0^t 2 M_u W_u \, dW_u + \int_0^t W_u^2 \, du.$$

We want $\mathbb{E}[M_t^2 - M_s^2 | \mathcal{F}_s]$

$$M_t^2 - M_s^2 = \int_s^t 2M_u W_u \, dW_u + \int_s^t W_u^2 \, du.$$

$$\mathbb{E}[M_t^2 - M_s^2 | \mathcal{F}_s] = \underbrace{\mathbb{E}\left[\int_s^t 2M_u W_u \, dW_u \,\Big|\, \mathcal{F}_s\right]}_{= 0} + \int_s^t \mathbb{E}[W_u^2 | \mathcal{F}_s] \, du.$$

$$\mathbb{E}\left[\int_0^t 2M_u W_u \, dW_u - \int_0^s 2M_u W_u \, dW_u \,\Big|\, \mathcal{F}_s\right].$$

$$= \int_0^s 2M_u W_u \, dW_u - \int_0^s 2M_u W_u \, dW_u = 0.$$

$$= \int_s^t \mathbb{E}[W_u^2 - u | \mathcal{F}_s] + u \, du.$$

$$= \int_s^t (W_s^2 - s + u) \, du = (W_s^2 - s)(t - s) + \frac{t^2 - s^2}{2}.$$

T or (F): $Z = X + Y$. Then

$$[Z,Z]_t = [X,X]_t + [Y,Y]_t.$$

$X_t = Y_t = W_t.$

$[X,X]_t = [Y,Y]_t = t. \to [X,X]_t + [Y,Y]_t = 2t.$

$Z_t = X_t + Y_t = 2W_t.$

$dZ_t = 0dt + 2dW_t.$

$d[Z,Z]_t = 4dt.$

$[Z,Z]_t = \int_0^t 4dt = 4t.$

If $dX_t = a_t\,dt + \sigma_t\,dW_t.$

$dY_t = b_t\,dt + \xi_t\,dW_t.$

$d(X+Y)_t \quad (a_t + b_t)dt + (\sigma_t + \xi_t)\,dW_t.$

$d[X+Y, X+Y]_t = (\sigma_t + \xi_t)^2\,dt.$

Common mistakes from last years. midterm

- $e^{XY} = e^X e^Y$. Not true.

$\mathbb{E}[e^X] = e^{\mathbb{E}[X]}$. Not true.

Be careful of calculation mistakes.

Δ_t is a function

$\int_0^t \Delta_t \, dW_t$ is a martingale.

$$dX_t = b_t \, dt + \sigma_t \, dW_t .$$

then $\int_0^t \Delta_t \, dX_t$ will not be a martingale.