Reminder: Midterm $\boxed{\text{One week from today}}$.

Ito's formula:

$$X \to \text{Itô process} = X(0) + \underbrace{\int_0^T b(t)\,dt}_{\substack{\text{R-int} \\ \text{(finite first variation)}}} + \underbrace{\int_0^T \sigma(t)\,dW(t)}_{\text{Itô integral}}.$$

This decomposition is called the

Itô decomposition (the semi-mg decomposition).

Notation: $dX(t) = b(t)\,dt + \sigma(t)\,dW(t)$.

(b, $\sigma \to$ adapted processes).

Compute $[X, X](T) = \int_0^T \sigma(t)^2 \, dt$.

$$X(T) = X(0) + B(T) + \cancel{k}\, M(T).$$

$Q.V.(B) = 0$

B has finite 1^{st} variation

$QV = \int_0^T \sigma(t)^2 \, dt.$

first variation $= \infty$.

Itô's formula: $X \longrightarrow$ Itô process. (semi_mg).

Let $f = f(t, x)$ = some non-random fn.

<u>Goal:</u> $f(t, X(t))$. ~~Itt~~ & decompose into BV part (Chain Rule).
+ Mg part.

① Suppose X is differentiable as a fn of t.
(this is NOT TRUE for most Itô processes).

Then ~~$f(t,x$~~

$$f(T, X(T)) - f(0, X(0)) = \int_0^T \partial_t \left[f(t, X(t)) \right] dt$$

(Notation $\partial_t = \dfrac{\partial}{\partial t}$).

$$\overset{\text{chain Rule}}{=\!=} \int_0^T \left[\partial_t f \Big|_{(t, X(t))} + \partial_x f \Big|_{(t, X(t))} \frac{dX}{dt} \right] dt .$$

$$= \int_0^T \partial_t f(t, X(t)) dt + \partial_x f(t, X(t)) \, dX(t)$$

$$\boxed{\partial_x^2 f = \frac{\partial^2 f}{\partial x^2}}$$

<u>Thm</u>: (Itô formula / Itô Doeblin formula).

If f is a $C^{1,2}$ function.

(i.e. Once continuously diff in t
twice conti diff in x) ← $\partial_t f$, $\partial_x f$ $\partial_x^2 f$ exist & are cts.

then $f(T, X(T)) - f(0, X(0)) =$

$$\int_0^T \partial_t f(t, X(t))\, dt + \int_0^T \partial_x f(t, X(t))\, dX(t)$$

$$+ \frac{1}{2}\int_0^T \partial_x^2 f(t, X(t))\, d[X, X](t).$$

Differential form:

$$d f(t, X(t)) = \partial_t f(t, X(t))\, dt + \partial_x f(t, X(t))\, dX(t)$$
$$+ \frac{1}{2} \partial_x^2 f(t, X(t))\, d[X, X](t).$$

Intuition: Simplify: Assume $f = f(x)$ (does not depend on t).

Assume $X = W$ (B.M.).

Itô's formula: $$f(W(T)) - f(W(0)) = \int_0^T f'(W(t))\, dW$$
$$+ \frac{1}{2} \int_0^T f''(W(t))\, dt.$$

$$f(W(T)) - f(W(0)) = \sum_{i=0}^{n-1} f(W(t_{i+1})) - f(W(t_i)).$$

$$\underset{\text{Taylor Expand}}{=} \sum_{i=0}^{n-1} f'(W(t_i))\Big(\overbrace{W(t_{i+1}) - W(t_i)}^{\Delta_i W}\Big)$$

$$+ \frac{1}{2} f''(W(t_i)) \left(W(t_{i+1}) - W(t_i)\right)^2$$

$$+ \text{Higher order terms.}$$

$$= \sum_{i=0}^{n-1} f'(W(t_i))(\Delta_i W) + \frac{1}{2}\sum_{i=0}^{n-1} f''(W(t_i))(\Delta_i W)^2$$

$$+ \text{H.O.T.}$$

$$\approx \int_0^T f'(W(t))\,dW + \frac{1}{2}\int_0^T f''(W(t))\,\underline{\underline{dt}}$$

Examples:

Eg 1: Compute the Q.V. of $W(t)^2$

Trick: Write $W(T)^2 - W(0)^2 = \int_0^T b(t)\,dt + \int_0^T \sigma(t)\,dW(t)$.

In this case $[W^2, W^2](T) = \int_0^T \sigma(t)^2\,dt$

$\boxed{\text{Task: Find } \sigma.}$

Use Ito: Let $f(t,x) = x^2$

$\partial_t f = 0, \quad \partial_x f = 2x, \quad \partial_x^2 f = 2.$

$$\Rightarrow d\left(W^2(t)\right) = \partial_t f \, dt + \partial_x f \, d\overbrace{X}^{W} + \frac{1}{2}\partial_x^2 f \, d\overbrace{[X, X]}^{dt}.$$

$$= 0 + 2W(t)\,dW(t) + \frac{1}{2}\cdot 2 \, dt$$

$$= 2W(t)\,dW(t) + dt.$$

$$\boxed{\Rightarrow [W^2, W^2](T) = \int_0^T 4W(t)^2 \, dt}$$

Eg: Let $M(t) = W(t)$ (BM) (M is a mg).

Let $N(t) = W(t)^2 - t$ (know N is a mg also).

Q: Is MN a mg?

Trick: Use Itô's fomula.

Goal: Write $M(T)N(T) - M(0)N(0) = \int_0^T b(t)\,dt + \int_0^T \sigma(t)\,dW(t)$

will only be a mg if $\boxed{b = 0}$

Lets compute :

$$\partial_t f = -x. \qquad \partial_x f = 3x^2 - t \qquad \partial_x^2 f = 6x.$$

$$M(t)N(t) = W(t)^3 - t\,W(t).$$

$$d(MN) = d\big(f(t, W(t))\big) \quad \text{where } f(t,x) = x^3 - tx.$$

$$= \partial_t f \, dt + \partial_x f \, dW + \tfrac{1}{2}\partial_x^2 f \, dt$$

$$= (-W(t))dt + (3W(t)^2 - t)\,dW + 3W(t)\,dt.$$

$$= [2W(t)]\,dt + \underbrace{[3W(t)^2 - t]\,dW}_{\text{this is a mg.}}.$$

$$\Rightarrow M(T)\,N(T) = 0 + \underbrace{\underbrace{\int_0^T 2W(t)\,dt}_{\text{Is NOT a mg.}} + \underbrace{\int_0^T \left(3W(t)^2 - t\right) dW}_{\text{Is a mg.}}}_{\text{Is NOT a mg.}}$$

Eg: Let $X(t) = t\sin(W(t))$.

Q: Is $X^2 - [X,X]$ a mg?

(Note If X is a mg then $X^2 - [X,X]$ is a mg.

If not $\longrightarrow$ dont know).

Sol: Let $f(t,x) = t\sin(x)$.

$$\partial_t f = \sin x$$

$$\partial_x f = t\cos x \qquad \partial_x^2 f = -t\sin x.$$

$$\Rightarrow dX = \partial_t f \, dt + \partial_x f \, dW + \frac{1}{2}\partial_x^2 f \, dt.$$

$$= \sin(W(t))\,dt + t\cos(W(t))\,dW - \frac{1}{2}t\sin(W(t))\,dt.$$

$$dX = \left[\sin(W(t)) - \frac{t}{2}\sin(W(t))\right]dt + t\cos(W(t))\,dW.$$

$$\boxed{\Rightarrow d[X,X](t) = t^2\cos^2(W(t))\,dt}$$

Now compute dX^2.

$f(t,x) = x^2$, $\partial_t f = 0$

$\partial_x f = 2x$

$\partial_x^2 f = 2.$

$$\Rightarrow dX^2 = \partial_t f\,dt + \partial_x f\,dX + \frac{1}{2}\partial_x^2 f\,d[X,X].$$

$$= 0 + 2X\left(\sin(W(t))\left(1 - \frac{t}{2}\right)dt + t\cos(W(t))\,dW\right).$$

$$+ \quad t^2 \cos^2(W(t)) \, dt.$$

$$dX^2 = 2X t \cos W(t) \, dW + \Big[2X \sin W(t) \left(1 - \tfrac{t}{2}\right) + t^2 \cos^2(W(t)) \Big] \, dt.$$

Q: Is $X^2 - [X, X]$ a mg.

$$d\left(X^2 - [X,X]\right) = 2X t \cos(W(t)) \, dW + \underbrace{2X \sin W(t) \left(1 - \tfrac{t}{2}\right)}_{\neq 0} . dt$$

$\Rightarrow X^2 - [X,X]$ is NOT a mg!

<u>Prop:</u> Suppose $M(t) = \int_0^t \sigma(s)\, dW(s)$. (Know M is a mg).

Then $M^2 - [M,M]$ is a mg.

<u>Pf:</u> Process: M. Fn: $f(x) = x^2$.

Ito: $d(M^2) = 0\, dt + 2M\, dM + \frac{1}{2} \cdot 2 \cdot d[M,M]$.

$$\Rightarrow d\left(M^2 - [M,M]\right) = \underbrace{2M\sigma\, dW}_{Mg} + 0\, dt.$$

$\Rightarrow M^2 - [M,M]$ IS a mg.

Compute $E\left(W(t)^2 \mid \mathcal{F}_s \right)$.

Use Ito: $d\left(W(t)^2\right) = 2W(t)\,dW + d[W,W]$.

$$= 2W\,dW + dt.$$

$$\Rightarrow W(t)^2 - W(s)^2 = \int_s^t 2W(r)\,dW(r) + \int_s^t dr$$

$$\Rightarrow W(t)^2 = W(s)^2 + \int_s^t 2W(r)\,dW(r) + t - s$$

$$\Rightarrow E\left(W(t)^2 \mid \mathcal{F}_s\right) = W(s)^2 + 0 + t - s.$$

Compute $E\left(W(t)^4 \mid \mathcal{F}_s\right)$.

Know: $d\, W(t)^4 = 4W(t)^3 dW + 6W(t)^2 dt$ by Itô.

$$\Rightarrow W(t)^4 = W(s)^4 + \int_s^t 4W(r)^3 dW(r) + \int_s^t 6W(r)^2 dr.$$

$$\Rightarrow E\left(W(t)^4 \mid \mathcal{F}_s\right) = W(s)^4 + 0 + E\left(\int_s^t 6W(r)^2 dr \,\Big|\, \mathcal{F}_s\right).$$

$$= W(s)^4 + \int_s^t E\left(6W(r)^2 \mid \mathcal{F}_s\right) dr.$$

$$= W(s)^4 + 6\int_s^t \left(W(s)^2 + (r-s)\right) dr.$$

$$= W(s)^4 + 6W(s)^2(t-s) + 3(t-s)^2$$

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