Recall: Last time:

Mg: $\longrightarrow$ Fair Game. $E(X_t \mid \mathcal{F}_s) = X_s$

Checked: BM is a Mg: $E(W(t) \mid \mathcal{F}_s) = W(s)$.

Quadratic Variation:

$$[X, X](T) = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} \left(X(t_{i+1}) - X(t_i) \right)^2$$

$\|P\| \to \max(t_{i+1} - t_i)$.

Compute $[W, W](T) = T$

Check: $W(T)^2 - [W,W]$ is a Mg.

Theorem 1: Let M be a cts mg with filt $\{\mathcal{F}_t\}$.

Then $E\left(M(t)^2\right) < \infty \iff E\,[M,M](t) < \infty$.

in this case $E\,M(t)^2 = E\,M(0)^2 + E\,[M,M](t)$

Moreover $M(t)^2 - [M,M](t)$ is a mg.

Thm 2: If A is a cts adapted increasing process with $A(0) = 0$

& M is a cts Mg.

Then If $M^2 - A$ is a mg $\Longrightarrow A = [M,M]$.

Intuition: If X has finite Q.V. $\Rightarrow$

$$(X(t+\delta t) - X(t))^2 \approx O(\delta t).$$

If X has finite first variation

$$\text{Expect } |X(t+\delta t) - X(t)| \approx O(\delta t).$$

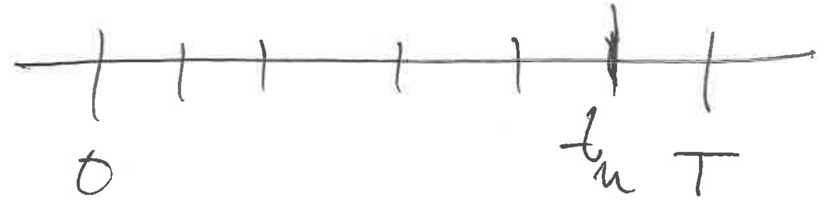
Construction of Ito Integral:

$W \to$ std BM. $\{\mathcal{F}_t\} \longrightarrow$ Brownian filtration.

Let D be an adapted process.

($D(t)$ = your position on an asset price is $W(t)$).

Let $P = \{0 = t_0 < t_1 \cdots < t_n \leq T\}$.



Assume we only trade at times t_k

Let $$I_P(T) = \sum_{i=0}^{n-1} D(t_i)\left(W(t_{i+1}) - W(t_i)\right). \cancel{+D(t_n)}$$

$$+ D(t_n)\left(W(T) - W(t_n)\right).$$

whenever $t_n \leq T < t_{n+1}$.

Lemma: (1) $$E\, I_p(T)^2 = E\left[\sum_{i=0}^{n-1} D(t_i)^2 (t_{i+1} - t_i) + D(t_n)^2 (T - t_n)\right]$$

if $T \in [t_n, t_{n+1})$.

(2) I_p is a martingale.

(3) $$[I_p, I_p](T) = \sum_{i=0}^{n-1} D(t_i)^2 (t_{i+1} - t_i) + D(t_n)^2 (T - t_n)$$

if $T \in [t_n, t_{n+1}]$.

Let's Check ①. (Assume $T = t_n$ for simplicity).

Notation: $\Delta_i W = W(t_{i+1}) - W(t_i)$.

$$I_p(T) = \sum D(t_i)\, \Delta_i W \qquad \text{if } T = t_n.$$

$$\Rightarrow E\left(I_p(T)\right)^2 = E\left(\sum_{i=0}^{n-1} D(t_i)\, \Delta_i W\right)^2.$$

$$= \underbrace{E \sum_{i=0}^{n-1} D(t_i)^2 (\Delta_i W)^2}_{①} + \underbrace{2 \sum_{j=0}^{n-1} \sum_{i=0}^{j-1} E\, D(t_i) D(t_j)\, \Delta_i W\, \Delta_j W}_{\text{Claim } = 0}.$$

②

Compute ①: $E\sum_{i=0}^{n-1} D(t_i)^2 (\delta_i W)^2$

$$= \sum_{i=0}^{n-1} E\left(D(t_i)^2 \left(W(t_{i+1}) - W(t_i)\right)^2\right).$$

$$= \sum_{i=0}^{n-1} E\, E\left(D(t_i)^2 \left(W(t_{i+1}) - W(t_i)\right)^2 \,\middle|\, \mathcal{F}_{t_i}\right).$$

$$= \sum_{i=0}^{n-1} E\left[D(t_i)^2\, E\left(\underbrace{\left(W(t_{i+1}) - W(t_i)\right)^2}\,\middle|\, \mathcal{F}_{t_i}\right)\right].$$

$$= \sum_{i=0}^{n-1} E\, D(t_i)^2 \left(t_{i+1} - t_i\right),$$

//

Check $E(2) = 0$: $\quad i < j$

$$\sum_{j=0}^{n-1} E\left(D(t_i) D(t_j)\, \Delta_i W\, \Delta_j W \right)$$

$$= E\, E\left(D(t_i) D(t_j)\, \Delta_j W\, \Delta_i W \,\middle|\, \mathcal{F}_{t_j} \right)$$

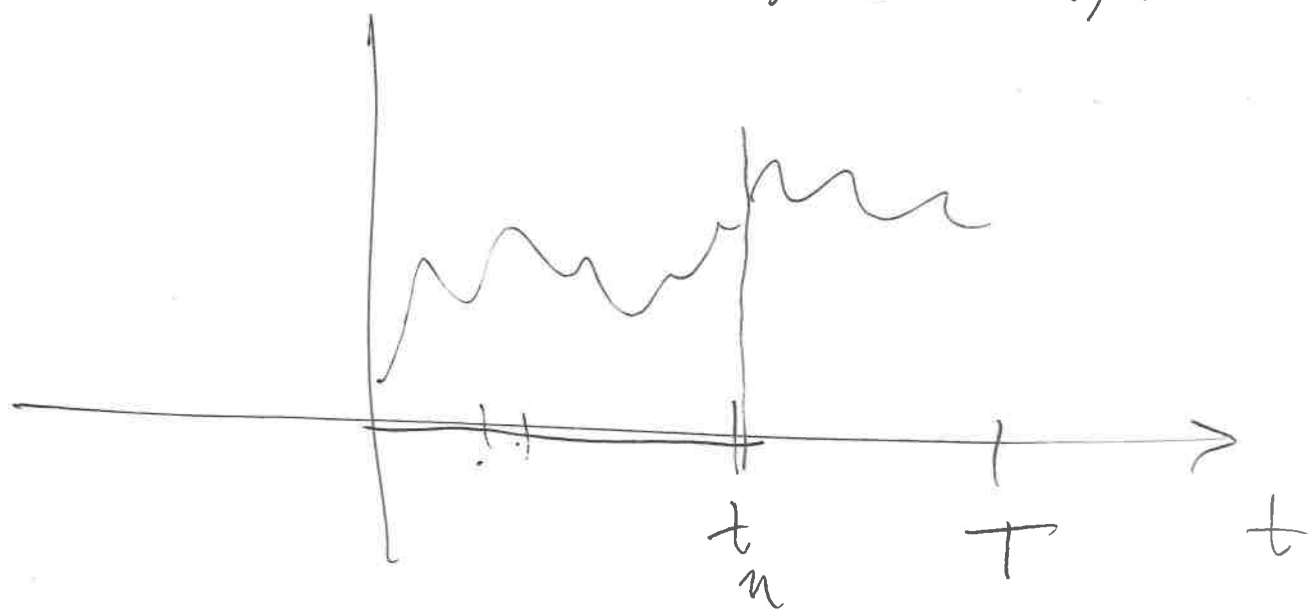
$$= E\left(D(t_i) D(t_j)\, \Delta_i W\, \underbrace{E\left(\Delta_j W \,\middle|\, \mathcal{F}_{t_j} \right)}_{0} \right)$$

$$= 0 \quad //$$

(2) You check I_p is a mg.

(3) $$I_p(T) = I_p(t_n) + D(t_n)\left(W(T) - W(t_n)\right).$$

$$\hookrightarrow [I_p, I_p](T) = [I_p, I_p](t_n) + D(t_n)^2 (T - t_n).$$



Itô's Idea:

Take partitions s.t. $\|P\| \longrightarrow 0$.

Note $$[I_P, I_P](T) = \sum_{i=0}^{n-1} D(t_i)^2 (t_{i+1} - t_i) + D(t_n)^2 (T - t_n).$$

Riemann Sum.

$$\lim_{\|P\| \to 0} [I_P, I_P](T) = \int_0^T D(t)^2 \, dt$$

Use this to show: the processes I_P themselves conge as $\|P\| \to 0$. The limit is the Itô integral.

Thm: If $\int_0^T D(t)^2\, dt < \infty$ almost surely.

the processes (I_P) conge as $\|P\| \to 0$

Let $I(T) = \lim_{\|P\| \to 0} I_P(T) = \int_0^T D(t)\, dW(t)$

$\hookrightarrow$ Itô integral.

Moreover If $E \int_0^T D(t)^2\, dt < \infty$, then

I_P is a mg &

$$[I, I](T) = \int_0^T D(t)^2\, dt$$

Remark: Key property needed was that D is adapted.

$$I_P(T) = \underbrace{\sum D(t_i)\left(W(t_{i+1}) - W(t_i)\right)}_{\text{looks like Riemann sum}}.$$

Rieman Int: OK to look at

$$\lim_{\|P\| \to 0} \sum D(\xi_i)\left(W(t_{i+1}) - W(t_i)\right)$$

where $\xi_i \in [t_i, t_{i+1}]$.

Will not work for Ito: In the above, to get ~~the~~ the Ito int need $\xi_i = t_i$.

Corollary (Ito Isometry).

If $E \int_0^T D(t)^2 \, dt < \infty$, then

$$E\left(\int_0^T D(t)\, dW(t)\right)^2 = E\int_0^T D(t)^2 \, dt.$$

Pf: Know $I(t) = \int_0^T D(t)\, dW(t)$ is a mg.

$I(0) = 0$

$$E\, I(T)^2 = E\, [I, I](T) = E\int_0^T D(t)^2 \, dt.$$

Goal: Itô's Formula (Generalization of Chain Rule).

Use ↑ to compute.

Let b & σ be 2 adapted processes.

Let $$X(t) = X(0) + \underbrace{\int_0^t b(s)\,ds}_{\text{Riemann Int}} + \underbrace{\int_0^t \sigma(s)\,dW(s)}_{\text{Itô integral}}.$$

Itô process: ① $X(0)$ non random

② $E\int_0^t \sigma(s)^2 ds < \infty$ & $\int_0^t b(s)\,ds < \infty$.

Short hand Notation for

$$X(t) = X(0) + \int_0^t b(s)\,ds + \int_0^t \sigma(s)\,dW(s).$$

& is

$$dX(t) = b(t)\,dt + \sigma(t)\,dW(t).$$

Q: What is $\cancel{[X,X](t)}$. $[X,X](T) = \int_0^t \sigma(s)^2\,ds.$

Prop: $dX = b\,dt + \sigma\,dW$.

Then $[X,X](T) = \int_0^T \sigma(s)^2\,ds$.

Proof: Let $B(t) = \int_0^t b(s)\,ds$. ← Bound variation (Finite first variation)

$M(t) = \int_0^t \sigma(s)\,dW(s)$. ← Mg part.

$$X(T) = X(0) + B(T) + M(T),$$

$\Delta_i X = X(t_{i+1}) - X(t_i)$, $\Delta_i B = B(t_{i+1}) - B(t_i)$

$\Delta_i M = M(t_{i+1}) - M(t_i)$.

$$\sum (\Delta_i X)^2 = \underbrace{\sum (\Delta_i M)^2}_{\text{converges to } [M,M]} + \underbrace{\sum (\Delta_i B)^2}_{\text{NTS} \to 0} + 2 \underbrace{\sum (\Delta_i M)(\Delta_i B)}_{\text{NTS} \to 0}.$$

$$= \int_0^T \sigma(s)^2 \, ds.$$

Note. $(\Delta_i B)^2 = \left(B(t_{i+1}) - B(t_i)\right)^2 = \left(\int_{t_i}^{t_{i+1}} b(s)\, ds\right)^2.$

$$\le (\max b)^2 (t_{i+1} - t_i)^2.$$

$$\Rightarrow \sum (\Delta_i B)^2 \le (\max b)^2 \sum (t_{i+1} - t_i)^2 \le (\max b)^2 \max (t_{i+1} - t_i) \cdot \sum (t_{i+1} - t_i)$$

$$\leq T (\max b)^2 \underbrace{\|P\|} \longrightarrow 0$$

You check: $\lim_{\|P\| \to 0} \sum (\Delta_i M)(\Delta_i B) \longrightarrow 0.$

Decomposition of X as a $mg^{(M)}$ + a process of finite first variation (B) is called the Itô decomposition. (& is unique).

If $dX = b\,dt + \sigma\,dW$,

& D is some adapted process, then

Riemann Int

we define

$$\int_0^T D(t)\,dX(t) = \int_0^T D(t)\,b(t)\,dt + \underbrace{\int_0^T D(t)\,\sigma(t)\,dW(t)}_{\text{Itô integral}}.$$

Suppose $dX = b\,dt + \sigma\,dW$.

$$\left(X(T) - X(0) = \int_0^T b(t)\,dt + \int_0^T \sigma(t)\,dW(t) \right).$$

Let $f = f(t, x)$ some non-random fn.

Let $Y(t) = f(t, X)$.

Guess. $$\boxed{dY = d\big(f(t,X)\big) = \partial_t f\,dt + \partial_x f\,dX + \frac{1}{2}\partial_x^2 f\,d[X,X].}$$