3 main topics:

- ↳ Cond Exp
- ↳ Brownian Motion
- ↳ Martingales.

Cond Exp:

sub σ - algebra.

$(\Omega, \mathcal{G}, \mathbb{P})$ $\qquad \mathcal{F} \subseteq \mathcal{G}$

X is a $\mathcal{G}$-meas RV.

$\mathbb{E}[X|\mathcal{F}]$ is the unique RV that satisfies

1). $\mathbb{E}[X|\mathcal{F}]$ is an $\mathcal{F}$-meas RV

2). "Partial Averaging"

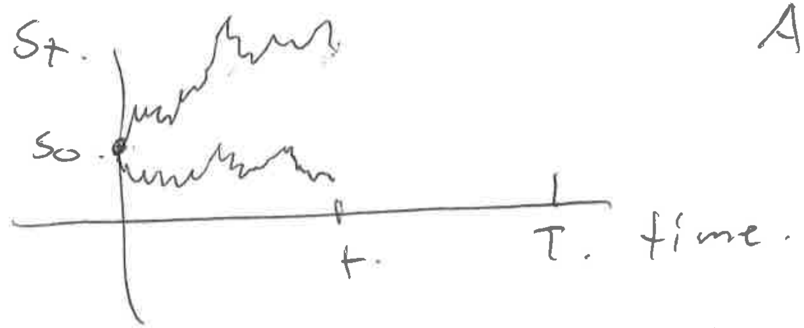
$$\mathbb{E}[X \mathbb{I}_A] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}] \mathbb{I}_A]$$

$$\forall A \in \mathcal{F}.$$

Example: X is payoff of some derivative security at time T. Fix $t < T$.

$\mathbb{E}[X|\mathcal{F}_t]$ is best approximation of X given our current knowledge.

$X(S_t)$.



$A = \{S_t > S_0\}$.

the new information is captured in $\mathbb{E}[X|\mathcal{F}_t]$

In this example $\mathbb{E}[X|\mathcal{F}_t]$ will depend on the value S_t or even the entire path $\{S_s : 0 \le s \le t\}$

so $\mathbb{E}[X|\mathcal{F}_t]$ is A $\boxed{\text{Random Variable}}$

Main connection: (Law of Total Expectation)

$$\mathbb{E}[\mathbb{E}[X|\mathcal{F}_t]] = \mathbb{E}[X].$$

Ex (Easiest to compute).

~~$(\Omega, \mathcal{F}, t$~~ $(\Omega, \mathcal{G}, \mathbb{P})$

$A, B \in \mathcal{G}$, $\mathbb{P}(B) \neq 0$

$\mathcal{F} = \sigma(\mathbb{1}_B) = \{\phi, B, B^c, \Omega\}$.

We want to show that

$$\mathbb{E}[\mathbb{1}_A \mid \sigma(\mathbb{1}_B)] = \underline{\mathbb{P}(A|B)\,\mathbb{1}_B + \mathbb{P}(A|B^c)\,\mathbb{1}_{B^c}}.$$

$\mathbb{P}(A|B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$.

Check (1) and (2) from definition.

(1) i.e. $\mathbb{P}(A|B)\,\mathbb{1}_B + \mathbb{P}(A|B^c)\,\mathbb{1}_{B^c}$ is $\mathcal{F}$-meas.

Notice. $\mathbb{E}[\mathbb{1}_A \mid \sigma(\mathbb{1}_B)](\omega) = \begin{cases} \mathbb{P}(A|B) \ ; & \omega \in B. \\ \mathbb{P}(A|B^c) \ ; & \omega \in B^c \end{cases}$

$c := \min\{\mathbb{P}(A|B), \mathbb{P}(A|B^c)\}$, $d := \max\{\mathbb{P}(A|B), \mathbb{P}(A|B^c)\}$.

Look at the sets $\mathbb{1}^{-1}(A)(-\infty, a) \in \mathcal{F}$ $\forall a \in \mathbb{R}$.

$$\mathbb{1}^{-1}(A)(-\infty, a) = \{w : \mathbb{1}(A) < a\}.$$

$X^{-1}(-\infty, a) = \{w : X(w) < a\}$

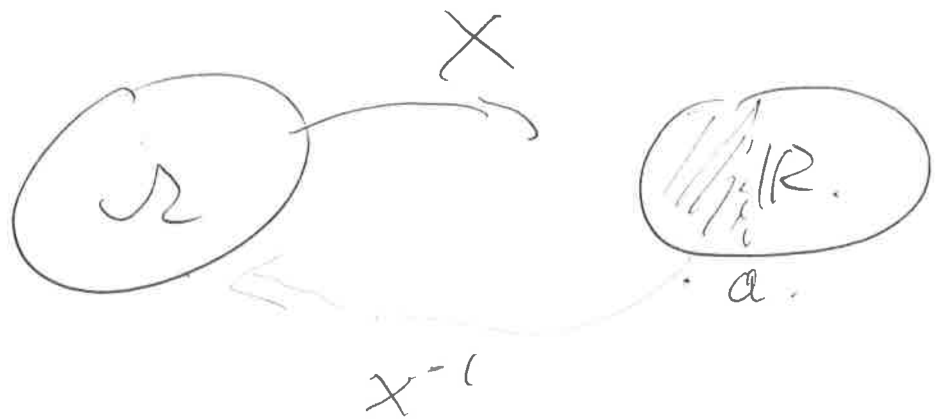
i) $a \le c$ then $\mathbb{1}^{-1}(A)(-\infty, a) = \emptyset \in \mathcal{F}$

ii) $c < a \le d$ then $\mathbb{1}^{-1}(A)(-\infty, a) = \begin{cases} B & \text{if } c = \mathbb{P}(A|B). \\ B^c & \text{if } c = \mathbb{P}(A|B^c). \end{cases}$ $\in \mathcal{F}$.

iii) $a > d$: ~~$\mathbb{1}(A)$~~ $\mathbb{1}^{-1}(A)(-\infty, a) = \Omega \in \mathcal{F}$ ✓.

(2). Need to check $\mathbb{E}[\mathbb{1}_A \mathbb{1}_F] = \mathbb{E}[(\mathbb{P}(A|B)\mathbb{1}_B + \mathbb{P}(A|B^c)\mathbb{1}_{B^c})\mathbb{1}_F]$

$\forall F \in \mathcal{F}$.



$F = \emptyset$: LHS $\mathbb{E}[\mathbb{1}_A \mathbb{1}_F] = \mathbb{E}[0] = 0$. $\checkmark$

RHS $= 0$ $\checkmark$

$F = \Omega$ LHS $\mathbb{E}[\mathbb{1}_A] = \mathbb{P}(A)$ $\checkmark$

RHS $\mathbb{E}[\mathbb{P}(A|B)\mathbb{1}_B + \mathbb{P}(A|B^c)\mathbb{1}_{B^c}]$.

$= \mathbb{P}(A|B)\,\mathbb{P}(B) + \mathbb{P}(A|B^c)\,\mathbb{P}(B^c)$.

$= \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) = \mathbb{P}(A)$ $\checkmark$

$F = B$ LHS $= \mathbb{E}[\mathbb{1}_A \mathbb{1}_B] = \mathbb{E}[\mathbb{1}_{A\cap B}] = \mathbb{P}(A\cap B)$.

RHS $= \mathbb{E}[(\mathbb{P}(A|B)\,\mathbb{1}_B + \mathbb{P}(A|B^c)\,\mathbb{1}_{B^c})\,\mathbb{1}_B]$.

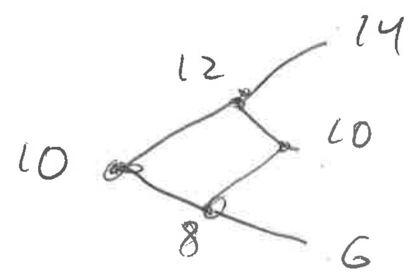
$$= \mathbb{E}[\mathbb{P}(A|B)\,\mathbb{1}_B^2 + \underbrace{\cancel{\mathbb{P}(A|B^c)\,\mathbb{1}_{B^c}\,\mathbb{1}_B}}_{0}].$$

$$= \mathbb{P}(A|B)\,\mathbb{E}[\mathbb{1}_B] = \mathbb{P}(A|B)\,\mathbb{P}(B) = \mathbb{P}(A\cap B) \checkmark$$

$\mathbb{1}_B = \mathbb{1}_B^2$

Check $F = B^c$.

From last week



Compute $\mathbb{E}\left[\frac{S_2 - 10}{10} \,\middle|\, \mathcal{F}_1\right]$.

This was not too hard for this example, but tedious!

Rules for cond. Exp.

1) Linearity, positivity.

$$\mathbb{E}[X + \alpha Y | \mathcal{F}] = \mathbb{E}[X|\mathcal{F}] + \alpha \mathbb{E}[Y|\mathcal{F}]$$

$X \leq Y$ then $\mathbb{E}[X|\mathcal{F}] \leq \mathbb{E}[Y|\mathcal{F}]$.

2) If X is $\mathcal{F}$-measurable then

$$\mathbb{E}[X|\mathcal{F}] = X.$$

If X is independent of $\mathcal{F}$ ($\perp\!\!\!\perp$)

then $\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X] \to$ constant.

3) (Took) X $\mathcal{F}$-meas, Y some RV

$$\mathbb{E}[XY|\mathcal{F}] = X\,\mathbb{E}[Y|\mathcal{F}]$$

4) (Tower) $\mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{G}$.

$$\mathbb{E}[X|\mathcal{E}] = \mathbb{E}\big[\mathbb{E}[X|\mathcal{F}]\,|\,\mathcal{E}\big].$$

5) Indep Lemma. X, Y RV's.

$X \perp\!\!\!\perp \mathcal{F}$ and Y is $\mathcal{F}$-meas

$f: \mathbb{R}^2 \to \mathbb{R}$.

then $$\mathbb{E}[f(X,Y)|\mathcal{F}] = \mathbb{E}^X[f(x,Y)] = g(Y).$$

If X has density p_X

$$\mathbb{E}[f(X,Y)|\mathcal{F}] = \int_{-\infty}^{\infty} f(x,Y)\,p_X(x)\,dx$$

<u>Ex</u> (Indep Lemma). X, Y are independent RV'S. $(\lambda > 0)$

for $\quad X \sim \exp(\lambda)$, $\quad Y \sim \text{unif}\,[0, \lambda]$

$$Z = f(X, Y) = e^{-XY^2}$$

Find $\mathbb{E}[Z \mid Y] = \mathbb{E}[Z \mid \sigma(Y)]$

<u>sol'n</u>

$$\mathbb{E}[Z|Y] = \mathbb{E}[e^{-XY^2} \mid Y] = \int_0^\infty e^{-xY^2} (\lambda e^{-\lambda x})\, dx$$

$$= \int_0^\infty \lambda e^{-x(\lambda + Y^2)}\, dx = \frac{-\lambda}{\lambda + Y^2} e^{-x(\lambda + Y^2)} \Big|_{x=0}^{x=\infty}$$

$$= \frac{\lambda}{\lambda + Y^2}$$

If we want to find $\mathbb{E}[Z]$

$$\mathbb{E}[Z] = \mathbb{E}[\mathbb{E}[Z \mid Y]] = \mathbb{E}\left[\frac{\lambda}{\lambda + Y^2}\right]$$

$$= \int_0^{\lambda} \frac{\lambda}{\lambda + y^2} \left(\frac{1}{\lambda}\right) dy.$$

$$= \frac{\arctan(\sqrt{\lambda})}{\sqrt{\lambda}}.$$

W_t

BM

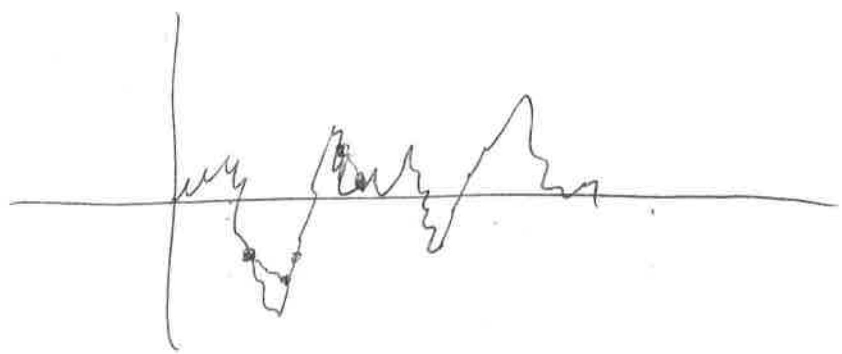
BM satisfies:

1) $W_0 = 0$.

2) $W_t \sim N(0, t)$

3) $s < t < u < v$ then $\underline{W_v - W_u \perp\!\!\!\perp W_t - W_s}$

(Independent Increments).



$\mathbb{E}[W_t | \mathcal{F}_s] = W_s$.

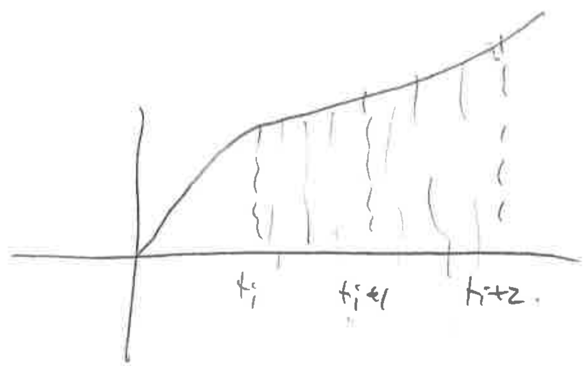
$\cancel{\Downarrow}$

Wrong. $\mathbb{E}[W_t] \neq 0$.

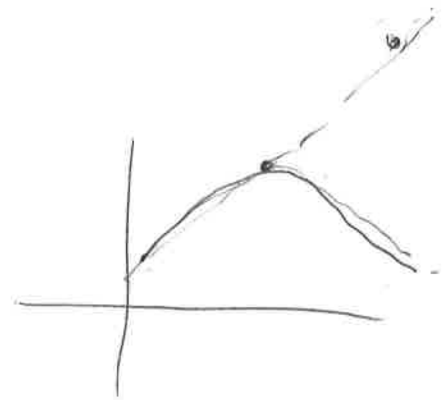
4) $t \mapsto W_t$ is continuous

(i.e. BM has continuous paths)

<u>Fact</u>: $V_W = \lim_{p \to 0} \sum_{i=1}^{n} |W_{t_{i+1}} - W_{t_i}| = \infty$ → $\{t_i\}$ partition of $[0,t]$



X



$$(X_{t_{i+2}} - \cancel{X_{t_{i+1}}}) + (\cancel{X_{t_{i+1}}} - X_{t_i}) = X_{t_{i+2}} - X_{t_i} < \infty$$

BM oscillates a lot.

partition on $[0,t]$

$$Q_W = \lim_{P \to 0} \sum_{i=1}^{n} (W_{t_{i+1}} - W_{t_i})^2 = t < \infty$$

why do we care?

If I trade ~~like~~ continuously like a ~~fund~~ fund often does. I want to keep track of my portfolio.

$\to 0 < t_1 < t_2 < \ldots < t_n = T$

If I trade at times $\{t_i\}_{i=1}^{n}$ only.

$\Delta_t \to$ # of shares of S I hold at time t.

$S_t \to$ price of S at time t.

my portfolio at T will be ~~...~~ $\sum_{i=1}^{n-1} \Delta_{t_i} (S_{t_{i+1}} - S_{t_i})$

continously we want $\lim_{P \to 0} \sum_{i=1}^{n-1} \Delta_{t_i} (S_{t_{i+1}} - S_{t_i})$ to exist.

If S_t is a function of BM it will not exist!

problem!!! To remedy this we will define the "stochastic Integral" which is different than Riemann or Lebesgue integration.

and it crucially uses QV.

<u>Simulate BM.</u> ✓.

1) Simulate BM $(T=1)$. $n=100$ first, $n=1000$ after

2) compute $\sum_{k=1}^{n} |W_k - W_{k-1}|$

↳ should get bigger with n.

$\sum_{k=1}^{n} (W_k - W_{k-1})^2$ ↳ should get close to T and will get closer as you increase n.

Martingales

↳ Most important type of process for us.

~~martingale~~ $(M_t)_{t \geq 0}$ is a martingale wrt $\mathcal{F}_t$.

if $\mathbb{E}[M_t | \mathcal{F}_s] = M_s \quad \forall\ s<t$.

and M_t is $\mathcal{F}_t$ measurable $\forall t$.

↳ Stochastic Integration

↳ stochastic integrals are martingales under some assumptions.

↳ R-N measures

T or F Martingales have constant expectations?

$$\mathbb{E}[M_t] = \mathbb{E}[\mathbb{E}[M_t \mid \mathcal{F}_s]] = \mathbb{E}[M_s]. \quad \forall s<t$$

$\therefore$ Constant expectation

Typically M_0 = constant and then

$\mathbb{E}[M_t] = M_0 . \quad \forall t.$

T or F stochastic processes with constant expectations are martingales?

<u>Ex</u>. W_t BM. $\mathcal{F}_t$ generated by W_t.

$\mathcal{F}_t = \sigma(W_s ; s \le t)$

$$X_t = \frac{1 + W_t^2}{1+t}$$

$$\mathbb{E}[X_t] = \frac{1 + \mathbb{E}[W_t^2]}{1+t} = 1 \rightarrow \text{constant expectation}$$

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = \frac{1 + \mathbb{E}[W_t^2 \mid \mathcal{F}_s]}{1+t}$$

$$\mathbb{E}[W_t^2 \mid \mathcal{F}_s] = \mathbb{E}[(W_t - W_s + W_s)^2 \mid \mathcal{F}_s]$$

$$= \mathbb{E}[(W_t - W_s)^2 \mid \mathcal{F}_s] + 2\,\mathbb{E}[(W_t - W_s) W_s \mid \mathcal{F}_s] + \mathbb{E}[W_s^2 \mid \mathcal{F}_s]$$

$$= \mathbb{E}[(W_t - W_s)^2] + 2 W_s \underbrace{\mathbb{E}[W_t - W_s]}_{0} + W_s^2$$

$$= t - s + W_s^2.$$

$\therefore$ ~~$\mathbb{E}$~~ $\mathbb{E}[X_t \mid \mathcal{F}_s] = \frac{1+(t-s+W_s^2)}{1+t}$.

since ~~$W_s^2 \neq s$ this is not~~

$s \neq t$ we do not have

$\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s$. So not a martingale.

From above calculation, we have

$\mathbb{E}[W_t^2 - t \mid \mathcal{F}_s] = W_s^2 - s$.

so $W_t^2 - t$ is a martingale!

if M is a martingale then

$M^2 - \underbrace{[M,M]}_{\text{QV}}$ is too.

T or F. If M and N are martingales is $M+N$ a martingale?

$$\mathbb{E}[M_t + N_t | \mathcal{F}_s] = \mathbb{E}[M_t | \mathcal{F}_s] + \mathbb{E}[N_t | \mathcal{F}_s].$$
$$= M_s + N_s.$$

T or F: What about MN.

NO Easiest example $M = N = W_t$.

You will soon define "quadratic covariation" between M and N. $[M,N]$.

and it will turn out that $MN - [M,N]$ will be a martingale.