Rules to compute:

(1) Independence lemma → Independent quantities — average
→ Measurable quantities → leave alone.

(Recitation → Examples)

Q: $E(E(X \mid \mathcal{G})) = EX$.

<u>Reason:</u> Know $E(\mathbb{1}_A X) = E(\mathbb{1}_A E(X \mid \mathcal{G}))$ for all $A \in \mathcal{G}$.

$\Omega \in \mathcal{G}$. Put $A = \Omega$,

Properties:

① $\alpha \in \mathbb{R}$ (non-random). $X, Y \to$ 2 RV's.

$$E(X + \alpha Y \mid \mathcal{F}) = E(X \mid \mathcal{F}) + \alpha E(Y \mid \mathcal{F}).$$

② If $X \leq Y$ a.s. then $E(X \mid \mathcal{F}) \leq E(Y \mid \mathcal{F})$.

③ If X is $\mathcal{F}$-measurable & Y is $\mathcal{G}$-meas.

$$E(XY \mid \mathcal{F}) = X E(Y \mid \mathcal{F})$$

④ (Tower Property) $\mathcal{E} \subseteq \mathcal{F} \subseteq \mathcal{G}$ are σ-alg.

$$E(X \mid \mathcal{E}) = E(E(X \mid \mathcal{F}) \mid \mathcal{E})$$

~~Exg~~ $W \longrightarrow$ std B.M.

<u>Brownian Filtration:</u> Pick $t \geq 0$.

$$\text{Let } \mathcal{F}_t = \sigma\Big(\bigcup_{s \leq t} \underbrace{\sigma(W(s))} \Big).$$

σ-alg of all events observable through $W(s)$.

Called the filtration generated by Brownian Motion.

More generally: X is any stochastic process.

$$\mathcal{F}_t^X = \sigma\Big(\bigcup_{s \leq t} \sigma(X(s)) \Big) \leftarrow \text{Filtration generated by } X.$$

Q: $s \leq t$: $\mathcal{F}_s^X \subseteq \mathcal{F}_t^X$

<u>Def</u>: A <u>filtration</u> $\{\mathcal{F}_t\}$ is a family of σ-alg. such that when $s \leq t$, $\mathcal{F}_s \subseteq \mathcal{F}_t$.

Intuition: $t \longrightarrow$ time.

$\mathcal{F}_t \longrightarrow$ information available up to time t.

<u>Def</u>: We say a process X is <u>adapted</u> to the filtration $\{\mathcal{F}_t\}$, if $X(t)$ is an $\mathcal{F}_t$ measurable R.V.

Let $W \longrightarrow$ std BM.

Let $\mathcal{F}_t = \mathcal{F}_t^W$ = Brownian filtration.

Compute $E(W(t) \mid \mathcal{F}_s)$ when $s < t$.

$$\hookrightarrow E(W(t) \mid \mathcal{F}_s) = E(W(t) - W(s) + W(s) \mid \mathcal{F}_s)$$

$$= \underbrace{E(W(t) - W(s) \mid \mathcal{F}_s)} + \underbrace{E(W(s) \mid \mathcal{F}_s)}.$$

$$= \underbrace{E(W(t) - W(s))}_{0} + W(s).$$

$$0 \quad + W(s)$$

//

Checked : $E(W(t) \mid \mathcal{F}_s) = W(s) \qquad (s < t).$

$\boxed{\text{Martingale Property}}$.

Def : A process X is called a martingale (mg) wrt the filtration $\{\mathcal{F}_t\}$ if

① X is adapted (i.e. $X(t)$ is $\mathcal{F}_t$ measurable).

& ② $E(X(t) \mid \mathcal{F}_s) = X(s)$. (Fair Game).

Sub matgale : $E(X(t) \mid \mathcal{F}_s) \geq X(s)$

Super mg : $E(X(t) \mid \mathcal{F}_s) \leq X(s)$.

Shown above: Brownian Motion is a Mg.

$\boxed{Q: \quad W(t)^2 \quad ?}$ ⟵ Eg's in Recitation.

Stochastic Integration:

$\Delta(t) \longrightarrow$ Your position at time t on some security.

$S(t) \longrightarrow$ Price of this security.

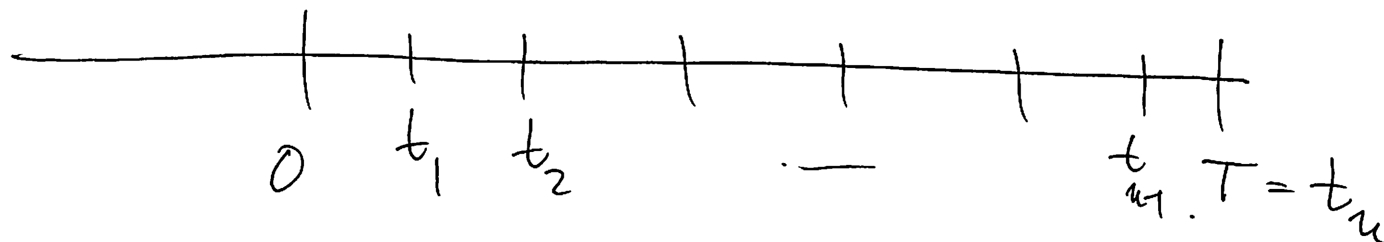
Say we only trade at times $0 = t_0 < t_1 < t_2 \cdots t_n = T$

Wealth up to time $T = t_n$, $X(T)$.

$$X(T) - X(0) = \sum_{i=0}^{n-1} \Delta(t_i) \left(S(t_{i+1}) - S(t_i) \right).$$

Allow trading continuously in time:

$$X(T) - X(0) = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} \Delta(t_i)\left(S(t_{i+1}) - S(t_i)\right).$$



$$P = \text{partition of } [0,T] = \{0, t_1, \dots t_n\}.$$

$$\|P\| = \max_i |t_{i+1} - t_i|.$$

looks like Riemann Integral of $\int_0^T \Delta(t)\, dS(t)$.

For this "limiting" procedure to work (essentially),

you need S to have "finite first variation".

Define, $V_{[0,T]}(S)$ = first variation of S on $[0,T]$.

$$= \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} |S(t_{i+1}) - S(t_i)|$$

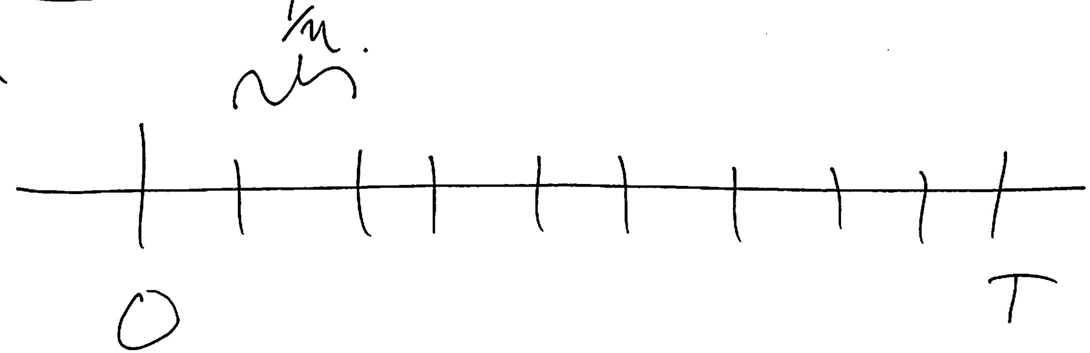
(If S is diff $V_{[0,T]}(S) = \int_0^T \left| \frac{dS}{dt} \right| dt$.)

Compute first variation of B. M.

$W \rightarrow$ std BM.

$$E\, V_{[0,T]}(W) = \lim_{\|P\| \to 0} E \sum |W(t_{i+1}) - W(t_i)|.$$

take a uniform partition



$$|t_{i+1} - t_i| = \frac{T}{n}.$$

$$\lim_{n \to \infty} E \sum_{i=0}^{n-1} \underbrace{|W(t_{i+1}) - W(t_i)|}_{\sim |N(0, \frac{T}{n})|}.$$

Know $E\left|N\left(0, \frac{T}{n}\right)\right| = c\sqrt{\frac{T}{n}}$ (c - some contant).

$$\Rightarrow E\, V_{[0,T]}(W) = \cancel{\lim_{\|P\|\to 0}} \lim_{n\to\infty} \underbrace{\sum_{i=0}^{n-1} c\sqrt{\frac{T}{n}}}$$

$$\lim_{n\to\infty} \sqrt{n} \cdot c\sqrt{T} = +\infty.$$

Try #2: Quadratic Variation.

<u>Def</u>: Let M be any process, we define the <u>Quadratic Variation</u> of M by.

$$[M,M](T) = \lim_{\|P\|\to 0} \sum_{i=0}^{n-1} \left(M(t_{i+1}) - M(t_i)\right)^2$$

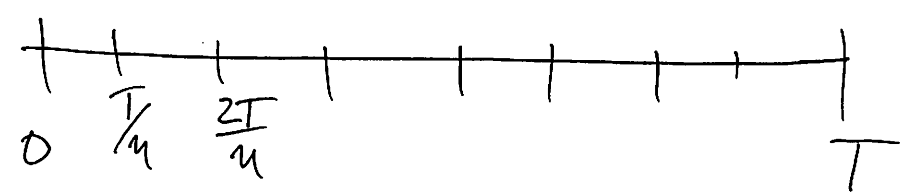
Theorem ~~Claim~~: If W is a std B.M. then

$$\boxed{[W,W](T) = T \quad \text{almost surely.}}$$

Proof: Notation: $\Delta_i W = W(t_{i+1}) - W(t_i)$

Simplify: Uniform partition: $t_{i+1} - t_i = \frac{T}{n}$

$$t_i = \frac{iT}{n}$$



Compute $\sum_{i=0}^{n-1} (\Delta_i W)^2 - T = \sum_{i=0}^{n-1} \left(W\left(\frac{(i+1)T}{n}\right) - W\left(\frac{iT}{n}\right) \right)^2 - T$

$$= \sum_{i=0}^{n-1} \xi_i$$

where $\xi_i = \left(W\left(\frac{(i+1)T}{n}\right) - W\left(\frac{iT}{n}\right) \right)^2 - \frac{T}{n}$.

Note ξ_i's i.i.d

$$\xi_i \sim N\left(0, \frac{T}{n}\right)^2 - \frac{T}{n}$$

$\Rightarrow E\xi_i = 0$ & $\mathrm{Var}\,\xi_i = \frac{T^2}{n^2}\left(E[N(0,1)^4] - 1\right)$.

$$\Rightarrow \sum_{i=0}^{n-1} \xi_i \quad \text{is mean } 0$$

$$\& \text{ variance } \sum_{i=0}^{n-1} Var(\xi_i) = n \frac{T^2}{n^2}\left(E\, N(0,1)^4 - 1\right).$$

$$\Rightarrow \lim_{n\to\infty} var\left(\sum_{i=0}^{n-1} \xi_i\right) = 0$$

$$\Rightarrow \lim_{n\to\infty} E\left(\sum_{i=0}^{n-1} (\Delta_i W)^2 - T\right) = 0$$

$$\boxed{\Rightarrow \lim_{n\to\infty} \sum_{i=0}^{n-1} (\Delta_i W)^2 = T.}$$

~~Cons~~ Claim: The process $W(t)^2 - [W,W](t)$ is a mg.

Check: $W(t)^2 - [W,W](t) = W(t)^2 - t$

Compute $E\left(W(t)^2 - t \,\middle|\, \mathcal{F}_s\right)$ & check this equals $W(s)^2 - s$.

Check: $E\left(W(t)^2 - t \,\middle|\, \mathcal{F}_s\right) = E\left(W(t)^2 \,\middle|\, \mathcal{F}_s\right) - t$

$= E\left(\left(W(t) - W(s) + W(s)\right)^2 \,\middle|\, \mathcal{F}_s\right) - t$

$= E\left(\left(W(t) - W(s)\right)^2 + W(s)^2 + 2\left(W(t) - W(s)\right) W(s) \,\middle|\, \mathcal{F}_s \right) - t$

$$= \left[E\,(W(t)-W(s))^2 \right] + W(s)^2 + 2W(s)\,E\left(W(t)-W(s) \mid \mathcal{F}_s\right) - t$$

$$= t - s + W(s)^2 + 0 - t$$

$$= W(s)^2 - s. \quad \Rightarrow W(t)^2 - t \text{ IS a } \text{Mg}.$$

Turns out: If M is any cts mg wrt $\{\mathcal{F}_t\}$.
Then $E\left[M(t)^2\right] < \infty \iff E\,[M,M](t) < \infty$.
& in this case $M^2 - [M,M]$ is a mg.