Independence of RV's.

① Events: A, B are independent if $P(A\cap B) = P(A)P(B)$.

② RV's. X, Y 2 RV's.

X & Y are ind if every event observable from X
is ind of " " " " Y.

Def: $\sigma(X) =$ All events observable from the RV X.

$= \sigma$-alg generated by $\{X \leq \alpha\}$ for all $\alpha \in \mathbb{R}$.

(Eg: $\{X \in (1,2)$ & $e^X < X^2 + 3\} \in \sigma(X)$.

Def: X & Y (2 RV's) are independent if for every $A \in \sigma(X)$ & $B \in \sigma(Y)$, A & B are independent.

Eg: $\Omega = \{1, 2, \cdots 52\}$.

Red cards $= R = \{1, \cdots 26\}$, $B = \{27, \cdots 52\}$.

Game: Red $\rightarrow +1$

Black $\rightarrow -1$

$X = \mathbb{1}_R - \mathbb{1}_B$. Compute $\sigma(X)$.

$$\left.\begin{array}{l} R = \{X = +1\} \in \sigma(X) \\ B = \{X = -1\} \in \sigma(X) \end{array}\right\} \longrightarrow \sigma(X) = \{R, B, \phi, \Omega\}.$$

Proposition: Let $X_1, \dots X_N$ be N random variables.

Then ① $X_1, \dots X_N$ are independat

$\Longleftrightarrow$ ② for every $\alpha_1, \dots \alpha_N \in \mathbb{R}$,

$$P(X_1 \le \alpha_1, X_2 \le \alpha_2, \dots X_N \le \alpha_N).$$

$$= P(X_1 \le \alpha_1)\, P(X_2 \le \alpha_2) \cdots P(X_N \le \alpha_N).$$

$\Longleftrightarrow$ ③ for every bdd cts fn's $f_1, f_2, \dots f_N$ (non-random).

$$E\left(f_1(X_1)\, f_2(X_2) \cdots f_N(X_N)\right) = \left(E f_1(X_1)\right)\left(E f_2(X_2)\right) \cdots \left(E f_N(X_N)\right).$$

$\iff$ (4) for every $t_1, t_2, \ldots t_N$ we have

Joint MGF of $(X_1, \ldots X_N)$.

$$E\left(\exp\left(t_1 X_1 + t_2 X_2 + \cdots t_N X_N\right)\right)$$

$$= \left(E \exp(t_1 X_1)\right)\left(E \exp(t_2 X_2)\right) \cdots \left(E \exp(t_N X_N)\right).$$

Compute covariance of BM.

$W \to$ standard Brownian Motion.

1. W is a cts process, & $W(0) = 0$.
2. Independent increments.
3. $W(t) - W(s) \sim N(0, t-s)$.

Compute $E\big(W(t)\,W(s)\big) = s \wedge t$ (s minimum t).

$\hookrightarrow$ Pf: ~~E(W~~ Suppose $s < t$

$$E\big(W(s)\,W(t)\big) = E\Big(W(s)\big(W(t) - W(s) + W(s)\big)\Big).$$

$$= E\left(\overset{W(s)}{\cancel{W(t)}}\,\big(W(t)-W(s)\big)\right) + \underbrace{E\,W(s)^2}_{s}.$$

independent

$$= \big(E\,W(s)\big)\Big(\underbrace{E\big(W(t)-W(s)\big)}_{0}\Big) + s.$$

$$= s.$$

Conditional Expectation

$(\Omega, \mathcal{G}, P)$.

Suppose $\mathcal{F} \subseteq \mathcal{G}$ is a σ- sub algebra of $\mathcal{G}$.

Suppose X is a $\mathcal{G}$-measurable random variable.

Conditional expectation of X (given $\mathcal{F}$)

is the "best approximation of X that is $\mathcal{F}$-measurable",

i.e. minimize $\{ E|X-Z|^2 \mid Z \text{ is a } \mathcal{F}\text{-meas RV} \}$.

Define $E(X \mid \mathcal{F}) =$ cond exp of the RV X given the σ-alg $\mathcal{F}$

$= $ the $\mathcal{F}$-meas RV, Z, that minimizes $E|X-Z|^2$

$= \arg\min \{ E|X-Z|^2 \mid Z \text{ is a } \mathcal{F} \text{ meas RV} \}$.

Eg: $\Omega = \{$ incomplete deck of cards.

30 cards

10 Red cards   20 Black cards.

5 High cards   5 low   4 High   16 low.

Game: High $\longrightarrow$ win \$1
Low $\longrightarrow$ lose \$1.

$$\left.\begin{array}{l}\text{High} \longrightarrow \text{win } \$1 \\ \text{Low} \longrightarrow \text{lose } \$1.\end{array}\right\} \quad X = \mathbb{1}_H - \mathbb{1}_L$$

"Stand too far" only see the <u>COLOR</u>.

$$\cancel{\mathcal{G}} \quad \mathcal{G} = \{\phi, \Omega, R, B, RH, RL, BH, BL\}.$$

& comb.

$$\mathcal{F} = \{\phi, \Omega, R, B\}.$$

$$E(X \mid \mathcal{F}) = (0)\,\mathbb{1}_R - \left(\frac{3}{5}\right)\mathbb{1}_B$$

For every event ~~like~~ $A \in \mathcal{F}$. $\qquad E(\mathbb{1}_A X)$

↳ Expected return of X (~~given~~ on A).

= Expected return of $E(X|\mathcal{F})$ (on A).

$E(\mathbb{1}_A E(X|\mathcal{F}))$.

$\boxed{\text{Probably def of conditional Expectation.}}$

Let X be a $\mathcal{G}$-meas RV.

Let $\mathcal{F} \subseteq \mathcal{G}$ be a σ-subalg.

Def: The cond exp of X given $\mathcal{F}$, $E(X|\mathcal{F})$, is a RANDOM VARIABLE such that

① $E(X \mid \mathcal{F})$ is $\mathcal{F}$ - measurable.

& ② for EVERY event $\boxed{A \in \mathcal{F}}$, we have

$$E\left(\mathbb{1}_A X\right) = E\left(\mathbb{1}_A E(X \mid \mathcal{F})\right).$$

$$\int_A X \, dP = \int_A E(X \mid \mathcal{F}) \, dP.$$

Prop 1: ① Suppose X is independent of $\mathcal{G}$

then $E(X \mid \mathcal{G}) = EX$.

② Suppose X is $\mathcal{G}$-measurable

then $E(X \mid \mathcal{G}) = X$.

Intuition on ①. Suppose $X = \sum a_i \mathbb{1}_{A_i}$.

Pick $B \in \mathcal{G}$ (X is independent of $\mathcal{G}$).

Compute $E(\mathbb{1}_B X) = E(\mathbb{1}_B \sum a_i \mathbb{1}_{A_i})$.

$$= E\left(\sum a_i \mathbb{1}_{A_i \cap B}\right).$$

$$= \sum a_i P(A_i \cap B) \overset{\text{independence}}{=\!=} \sum a_i P(A_i) P(B).$$

$$= P(B)\left(\sum a_i P(A_i)\right).$$

$$= P(B)\, EX$$

$$= E\left(\mathbb{1}_B (EX)\right) = \underbrace{E\left(\overbrace{E(X|\mathcal{F})}^{(EX)} \cdot \mathbb{1}_B\right)}.$$

$$\Rightarrow E(X \mid \mathcal{G}) = EX$$

(Only happens because X is ind of $\mathcal{G}$).

$\boxed{\text{Independence Lemma:}}$ X & Y are two RV's.

$X \longrightarrow$ independent of $\mathcal{G}$

$Y \longrightarrow$ $\mathcal{G}$ meas.

Let $f = f(x, y)$ be some non-random function.

$$E(f(X, Y) \mid \mathcal{G}) = g(Y) \text{ where}$$

$g = g(y)$ is the non-random funtion defined by.

$$g(y) = \mathbb{E}\, f(X, y).$$