Stochastic Calculus for Finance I

TA: David Itkin

Office: WEH 7211.

email: ditkin@andrew.cmu.edu.

OH. PITT: WED 4:30 - 5:30

NYC: WED 2:30 – 3:30.

Canvas.

Let me know in advance.

2 parts.

Main Goal: To develop continuous time Financial Models.

Math.

- Stochastic Integral
- Ito's Lemma
- Girsanov's Theorem.

Black-Scholes Model. + extensions.

Risk-neutral pricing.

Today:

- σ-algebras, RV's, probability measures and independence

<u>Recall:</u> $\mathcal{F}$ set of subsets that satisfies.

i) $\emptyset \in \mathcal{F}$

ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.

iii) $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F} \Rightarrow \bigcup_n A_n \in \mathcal{F}$

is called a <u>σ-algebra</u>.

For us σ-algebra's will represent <u>information</u>

T or F   If $\mathcal{F}$ and $\mathcal{G}$ are σ-algebras is $\mathcal{F} \cap \mathcal{G}$ a σ-algebra?

i) $\phi \in \mathcal{F}$, $\phi \in \mathcal{G} \Rightarrow \phi \in \mathcal{F} \cap \mathcal{G}$.

ii). $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$; $A \in \mathcal{G} \Rightarrow A^c \in \mathcal{G}$.

$$A \in \mathcal{F} \cap \mathcal{G} \Rightarrow A^c \in \mathcal{F} \cap \mathcal{G}.$$

iii). $\{A_n\}_{n \geq 1}^{\infty} \in \mathcal{F} \cap \mathcal{G} \Rightarrow \bigcup_n A_n \in \mathcal{F}$ and $\bigcup_n A_n \in \mathcal{G}$ ✓

T or F. If $\mathcal{F}$ and $\mathcal{G}$ are σ-algebras is $\mathcal{F} \cup \mathcal{G}$ a σ-algebra?

condition iii) can fail

EX: (coin Toss) $\Omega = \{(HH), (HT), (TH), (TT)\}$.

$$\mathcal{F} = \sigma(\{HH\}) = \{\phi, \Omega, \{HH\}, \{(HT), (TH), (TT)\}\}.$$

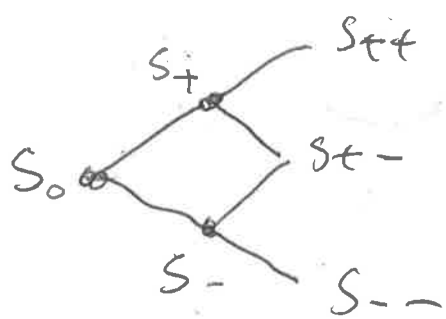
$$\mathcal{G} = \sigma(\{TT\}) = \{\phi, \Omega, \{TT\}, \{(HH), (HT), (TH)\}\}$$

$\{HH\}, \{TT\} \in \mathcal{F} \cup \mathcal{G}$.

<u>but</u> $\{(HH), (TT)\} \notin \mathcal{F} \cup \mathcal{G}$.

In fact $\sigma(\mathcal{F} \cup \mathcal{G}) = \{\phi, \Omega, \{HH\}, \{TT\}, \{TT, HT, HH\}, \{TT\},$
$\{TT, HT, TH\}, \{TT, HT, TH\}, \{\{HH\}, \{TT\}\},$
$\{HT, TH\}\}$.

<u>Ex</u> (Binomial Tree) → discrete.



$\Omega = \{$ all possible stock price paths $\}$

$= \{(S_+, S_{++}), (S_+, S_{+-}), (S_-, S_{+-}),$
$(S_-, S_{--})\}$.

$\mathcal{F}_0 = \{\phi, \Omega\}$. $\to$ info at time 0.

$\mathcal{F}_1 = \{\phi, \Omega, \{(S_+, S_{++}), (S_+, S_{+-})\}, \{(S_-, S_{+-}), (S_-, S_{--})\}\}$.

$\hookrightarrow$ info at time 1.

$\mathcal{F}_2 = \mathcal{P}(\Omega)$ = all subsets.

Random Variables $(\Omega, \mathcal{F}, \mathbb{P})$.

We say a function $X : \Omega \to \mathbb{R}$ is a Random Variable if. $\forall a \in \mathbb{R}$ $\{\omega \in \Omega : X(\omega) < a\} \in \mathcal{F}$.

$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad$ $\| \; X^{-1}(-\infty, a)$

" X is a RV if $\forall$ real number a you can find the set of outcomes that make $X < a$ in the σ-algebra $\mathcal{F}$ "

1) $\Omega = \{$ all outcomes of N coin tosses $\}$.

$X = \#$ of Heads. Then X is a RV wrt

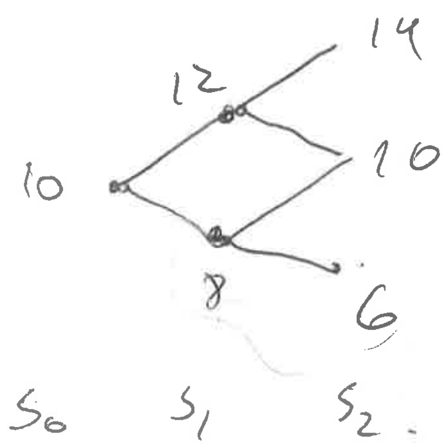
$\mathcal{F} = \mathcal{P}(\Omega)$.

2) $\Omega = \{(1,1), (1,2), \dots, (1,6), \dots, (6,6)\}$.

$\mathcal{F} = \mathcal{P}(\Omega)$

$X =$ sum of dice throws is a RV.

3)



$\Omega = \{(12,14), (12,10), (8,10), (8,6)\}$.

$\mathcal{F}_1$ and $\mathcal{F}_2$ as above.

We are interested in stock price returns.

define $X_1(\omega) = \dfrac{S_1(\omega) - 10}{10}$

Q1: Is this a RV wrt $\mathcal{F}_1$?

Yes: Notice X takes on 2 values

$$X = \begin{cases} \frac{1}{5} & \text{if } S_1 = 12 \\ -\frac{1}{5} & \text{if } S_1 = 8 \end{cases}$$

if: $a \le -\frac{1}{5}$: $X^{-1}(-\infty, a) = \phi \in \mathcal{F}_1$ ✓

$-\frac{1}{5} < a \le \frac{1}{5}$: $X^{-1}(-\infty, a) = \{(8,10), (8,6)\} \in \mathcal{F}_1$. ✓

$a > \frac{1}{5}$: $X^{-1}(-\infty, a) = \Omega \in \mathcal{F}_1$ ✓

This verifies that ~~it~~ is a RV. wrt $\mathcal{F}_1$.

Q2: Is X_1 a RV wrt $\mathcal{F}_2$?

<u>yes</u>:

$\forall a \in \mathbb{R}$ $\quad X^{-1}(-\infty, a) \in \mathcal{F}_1 \subseteq \mathcal{F}_2$.

In general if $\mathcal{F} \subseteq \mathcal{G}$ are σ-algebras and X is a RV wrt $\mathcal{F}$ then it is also a RV wrt $\mathcal{G}$.

<u>Q3:</u> Define $X_2(\omega) = \dfrac{S_2(\omega) - 10}{10}$.

Is this a $\mathcal{F}_1$-RV?

<u>NO</u>.

Notice $X_2 = \frac{2}{5}$ or $X_2 = 0$ or $X_2 = -\frac{2}{5}$.

take $a \in (-\frac{2}{5}, 0)$.

$X^{-1}(-\infty, a) = \{(8,6)\} \notin \mathcal{F}_1$.

So not a RV. wrt $\mathcal{F}_1$.

You can check X_2 is a RV wrt $\mathcal{F}_2$ ✓.

Probabilities Given $(\Omega, \mathcal{F})$ we want to assign weights to events $\to$ these weights are called probabilities.

Def $\mathbb{P} : \mathcal{F} \to [0,1]$. is a probability ~~measure~~ if it satisfies

i) $\mathbb{P}(\Omega) = 1$, ii) $\{A_n\}_{n\geq 1}^{\infty} \in \mathcal{F}$ are pairwise disjoint then $\mathbb{P}(\bigcup_n A_n) = \sum_n \mathbb{P}(A_n)$.

Back to binomial tree.

If we let $p \in [0,1]$ be given.

and we define $\mathbb{P} : \mathcal{F}_1 \to [0,1]$

$$\mathbb{P}(\phi) = 0, \quad \mathbb{P}(\Omega) = 1, \quad \mathbb{P}(\{(12,14),(12,10)\}) = p$$

$$\cancel{\mathbb{P}(\{(8}, \quad \mathbb{P}(\{(8,6),(8,10)\}) = 1-p.$$

we have that $Z = \begin{cases} 1 & \text{if } S_1 = 12 \\ 0 & \text{if } S_1 = 8 \end{cases}$

is a Bernoulli RV with prob of success p.

In this course we will ~~use~~ develop continuous-time models for asset pricing. time will be $t \in [0,T]$ T is fixed

and $S_t \in \mathbb{R}^+ = [0,\infty)$.

Advantages: - mimics real trading more

- you can do more analytically.

Disadvantages: - math is harder

- discrete time models are easier to "code up" on your computer.

---

typically "information flow" (i.e. σ-algebras) will be impossible to write down explicitly.

Ex (Borel σ-algebra). Most important σ-algebra on $\mathbb{R}$ or $[0,1]$.

$\mathcal{B}$ = smallest σ-algebra containing $(-\infty, a)$ $\forall a \in \mathbb{R}$.

(a, ∞)

(a, b)

prove ~~Exc~~ for $a \le b$ $[a,b] \in \mathcal{B}$.

$\forall n \in \mathbb{N}.\ (-\infty, b + \frac{1}{n}) \in \mathcal{B}$

~~Bys~~ $\cancel{\left(\bigcup_n (-\infty, b+\frac{1}{n})\right) \in \mathcal{B}}$

$\cancel{= \bigcap_n (-\infty, b+\frac{1}{n}) \in \mathcal{B}}$

$\bigcup_n (-\infty, b+\frac{1}{n}) \in \mathcal{B}$ by 2, but you can also show $\bigcap_n (-\infty, b+\frac{1}{n}) \in \mathcal{B}$.

$$\bigcap_n (-\infty, b+\tfrac{1}{n}) = (-\infty, b] \in \mathcal{B}.$$

$(-\infty, a) \in \mathcal{B} \Rightarrow [a, \infty) \in \mathcal{B}$.

$\mathcal{B} \ni [a,\infty) \cap (-\infty, b] = [a,b]$.

stuff like $\bigcup_n \bigcap_m \bigcup_k [a - \frac{1}{n} + k^{-m}, b] \in \mathcal{B}$.

much harder to categorize $\mathcal{B}$!

Independence : $(\Omega, \mathcal{F}, \mathbb{P})$

$\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathcal{F}$ σ-algebras.

$\mathcal{F}_1$ and $\mathcal{F}_2$ are independent σ-algebras if

$\forall A \in \mathcal{F}_1, B \in \mathcal{F}_2 \quad \mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$.

if X and Y are RV's we say they are independent if $\sigma(X)$ and $\sigma(Y)$ are.

$$\sigma(X) = \sigma(\{X \leq a\} ; a \in \mathbb{R}).$$

X, Y are independent $\iff$ $\underbrace{f_{X,Y}}_{\text{joint density}}(x,y) = f_X(x)\, f_Y(y)$.

Ex. X, ~~are~~ is a positive-integer valued. RV.

(i.e. $X(\omega) \in \{1, 2, 3, \dots\}$.)

$$\mathbb{P}(X = n) = \frac{n^{-2}}{\sum_n n^{-2}} = \frac{n^{-2}}{c}$$

$E_2 = \{X$ is divisible by $2\}$, $E_3 = \{X$ is divisible by $3\}$.

Show that E_2 and E_3 are independent events.

i.e. show $\mathbb{P}(E_2 \cap E_3) = \mathbb{P}(E_2)\,\mathbb{P}(E_3)$.

$$\mathbb{P}(E_2) = \mathbb{P}(X = 2 \text{ or } X = 4 \text{ or } X = 6 \dots)$$

$$= \mathbb{P}\Big(\bigcup_{m \in \mathbb{N}} \{X = 2m\}\Big) = \sum_{m=1}^{\infty} \mathbb{P}(X = 2m)$$

$$= \sum_{m=1}^{\infty} \frac{(2m)^{-2}}{c} = 2^{-2} \frac{\sum_{m=1}^{\infty} m^{-2}}{\sum_{n=1}^{\infty} n^{-2}} = \frac{1}{4}.$$

$$\mathbb{P}(E_3) = \mathbb{P}\left(\bigcup_{m=1}^{\infty} \{X = 3m\}\right) = \sum_{m=1}^{\infty} \mathbb{P}(X = 3m).$$

$$= \sum_{m=1}^{\infty} \frac{(3m)^{-2}}{c} = 3^{-2} = \frac{1}{9}.$$

$$\mathbb{P}(E_2 \cap E_3) = \mathbb{P}(X \text{ is divisible by } 6).$$

$$= \mathbb{P}\left(\bigcup_{m=1}^{\infty} \{X = 6m\}\right)$$

$$= \sum_{m=1}^{\infty} \mathbb{P}(X = 6m) = \sum_{m=1}^{\infty} \frac{(6m)^{-2}}{c} = 6^{-2} = \frac{1}{36}.$$

indeed $\mathbb{P}(E_2 \cap E_3) = \frac{1}{36} = \frac{1}{9} \cdot \frac{1}{4} = \mathbb{P}(E_2)\,\mathbb{P}(E_3)$ ✓

Hint for HWK: Prob 4(b)

If g is an invertible function (and differentiable)

then $\frac{d}{dx} g^{-1}(x) = \frac{1}{g'(g^{-1}(x))}$.

for part (a)

Try to rewrite it as a double integral and switch order of integration.