Last time: Models for stock prices.

$$\hookrightarrow \; dS = \alpha S \, dt + \underbrace{\sigma S \, d\tilde{W}}_{\text{noisy fluctuations}}$$

B.M.

① Probability Space $(\Omega, \mathcal{G}, P)$

Ω: sample space (non-empty set).

$\mathcal{G}$: ~~σ-alg~~ σ-algebra

Represents events of which the probability is known.

$$\mathcal{G} = \{ \text{events } A \mid \quad \}. \qquad \mathcal{G} \subseteq \mathcal{P}(\Omega).$$

$P \longrightarrow$ Probability measure.

For every event A, $P(A) \in [0, 1]$

represents the probability of A occurring.

(for every $A \in \mathcal{G}$, $P(A) \in [0, 1]$).

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1. $\mathcal{G} \rightarrow$ closed under complements. $(A \in \mathcal{G} \Rightarrow A^c \in \mathcal{G})$.
2. $\mathcal{G}$ is closed under countable unions.

(i.e. If $A_1, A_2, A_3, \ldots \in \mathcal{G} \Longrightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{G}$).

(A σ-alg is a non empty collection of sets that satisfies 1 & 2)

(3) $\phi \in \mathcal{G}$ & $\Omega \in \mathcal{G}$

(4) If $A_1, \ldots \in \mathcal{G}$ then $\bigcap_{i=1}^{\infty} A_i \in \mathcal{G}$

(5) If $A, B \in \mathcal{G}$ then $A - B \in \mathcal{G}$.

$\left(\because\ A - B = A \cap B^c \right)$

P : Probability measure.

(1) For every $A \in \mathcal{G}$, $P(A) \in [0, 1]$, $P(\phi) = 0$ & $P(\Omega) = 1$.

(2) If $A_1, A_2, \ldots \in \mathcal{G}$ and are PAIRWISE DISJOINT

(i.e. $A_i \cap A_j = \phi$ when $i \neq j$).

Then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$.

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① & ② is the definition of a probability measure.

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③ $P(A^c) = 1 - P(A)$.

④ If $A_1, A_2, \ldots$ is an increasing seq of events.

$$A_1 \subseteq A_2 \subseteq A_3 \cdots\cdots$$

Then $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{i \to \infty} P(A_i)$.

~~5 If~~ 5 If $A_1 \supseteq A_2 \supseteq A_3 \cdots$ then

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) = \cancel{\text{limit}}\ \lim_{i \to \infty} P(A_i).$$

Random Variables: A random variable is a fn $X : \Omega \longrightarrow \mathbb{R}$.

Observe the event $\{X > 0\}$.

$$\{\omega \in \Omega \mid X(\omega) > 0\}$$

Can we compute $P(\{X > 0\})$? $(= P(X > 0))$.

Yes, provided $\{X > 0\} \in \mathcal{G}$

Require: for X to be a random variable.

then • "all questions" about X will lead only to events in $\mathcal{G}$.

Eg: $\{X > \alpha\} \in \mathcal{G}$ for every $\alpha \in \mathbb{R}$.

OR $\{X \in [a, b)\} \in \mathcal{G}$ " " $a, b \in \mathbb{R}$.

OR $\{e^{X+10} < \sin(5X)\} \in \mathcal{G}$.

Def: We say X is a $\mathcal{G}$-measurable random variable if for every $\alpha \in \mathbb{R}$, $\{X \leq \alpha\} \in \mathcal{G}$.

Remark: If X & Y are 2 random variables.

then $X+Y$, $X-Y$, $X+\alpha Y$ $(\alpha \in \mathbb{R})$, XY, $\frac{X}{Y}$

are all random variables.

Also, If f is any (non-random) fn

(Eg $f(x) = x^2$,
$f(x) = \ln(3x-5)$) & X is a R.V.

$$f(X) = f \circ X \text{ is also a RV.}$$

Eg: Say $A \subseteq \Omega$.

Define $\mathbb{1}_A$ ($\leftarrow$ fn with domain Ω & target $\mathbb{R}$).

$$\hookrightarrow \mathbb{1}_A \; \cancel{=} \; \mathbb{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A. \end{cases}$$

(indicator fn of A).

Q: Is $\mathbb{1}_A$ a random variable?

$$\mathbb{1}_A \text{ is a RV} \iff A \in \mathcal{G}.$$

Eg 2: Say $A_1, A_2 \ldots A_M$ are M events in $\mathcal{G}$.

Let $a_1, a_2 \ldots a_m \in \mathbb{R}$.

Define $$X = \sum_{i=1}^{M} a_i \mathbb{1}_{A_i}$$

Such random variables are called <u>SIMPLE</u> RANDOM VARIABLES.

Expectations: The Expectation of a random variable is the "<u>Lebesgue integral</u>". (measures the "Expected reward").

If X is a RV,

Notation for Expatrion of X is

$$EX = \int_\Omega X \, dP$$

Say X is simple: $X = \sum_{i=1}^{M} a_i \mathbb{1}_{A_i}$

then $EX = \sum_{i=1}^{M} a_i P(A_i) = \sum_{i=1}^{M} a_i \left(P(X = a_i) \right)$.

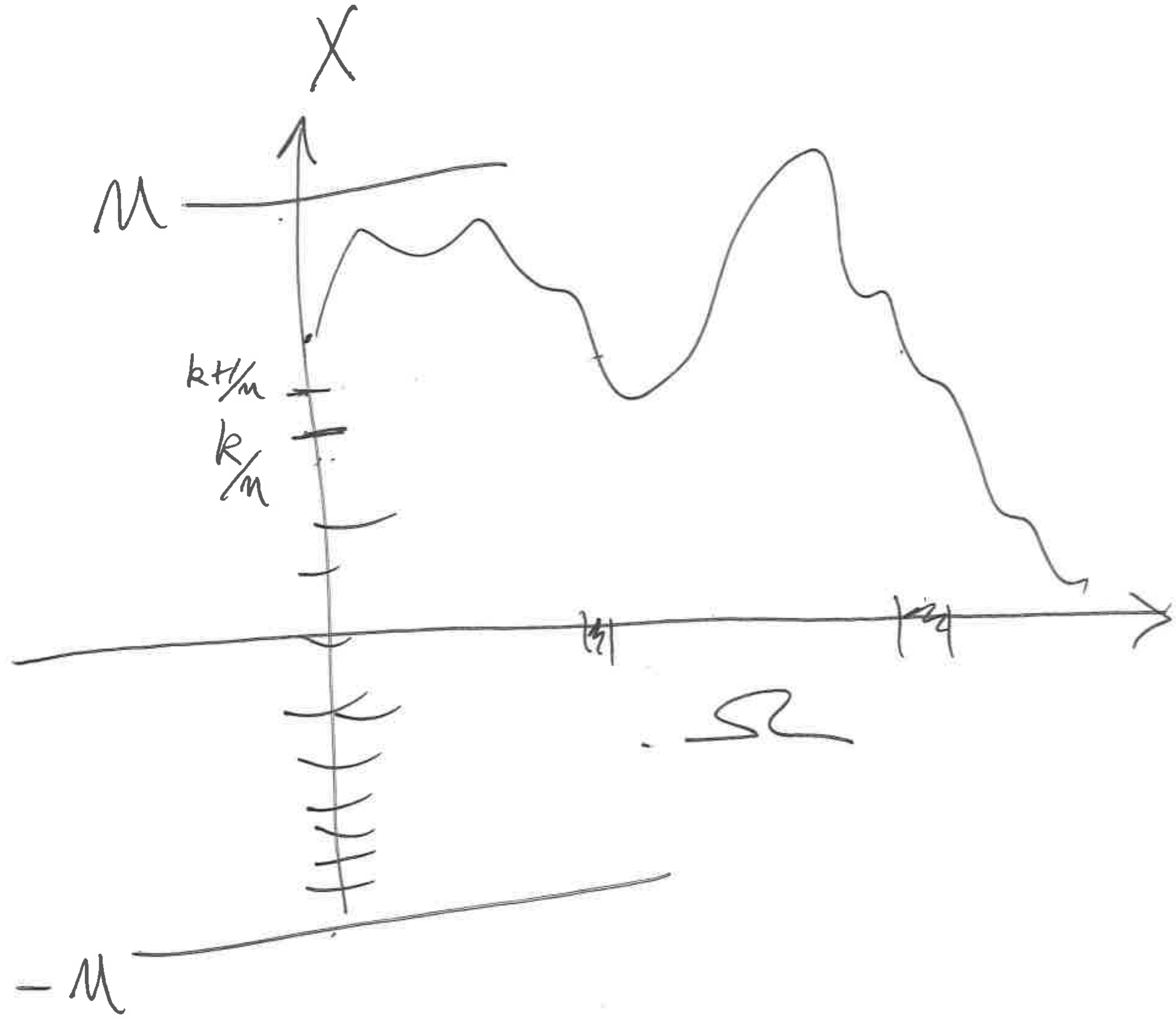
(if A_i's are disj).

If X is not simple, approximate X by simple RV's & take lmts.

Define $$EX = \lim_{n\to\infty} E\left(\sum_{-n^2}^{n^2-1} \frac{k}{n} \mathbb{1}_{\left\{ \frac{k}{n} \le X \le \frac{k+1}{n} \right\}} \right).$$

$\nearrow$

Gives the Lebesgue integral.


X
M
k+1/n
k/n
-M
Ω

To work with expectations, going to use the following properties:

① Linearity: If $\alpha \in \mathbb{R}$, X, Y are RV's.

then $E(X + \alpha Y) = EX + \alpha EY$.

② If $X \geq 0$ almost surely (i.e. $P(X \geq 0) = 1$).

Then $EX \geq 0$.

Also if additionally $P(X > 0) > 0$, then $EX > 0$.

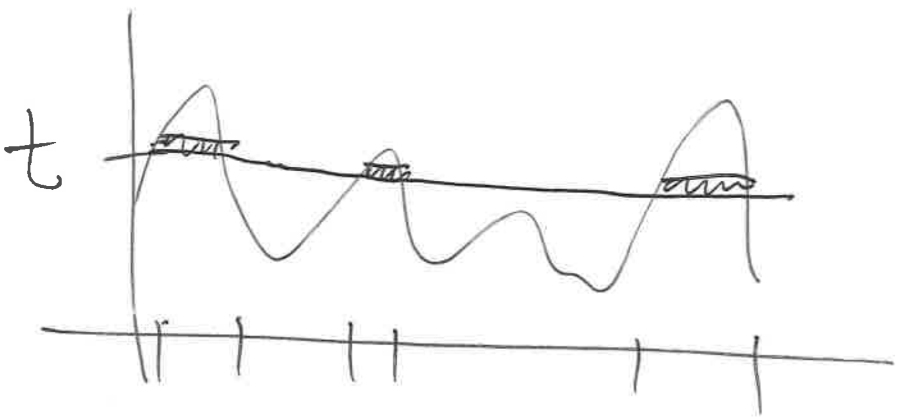
$\Rightarrow$ If $X \leq Y$ almost surely $\Rightarrow$ $EX \leq EY$

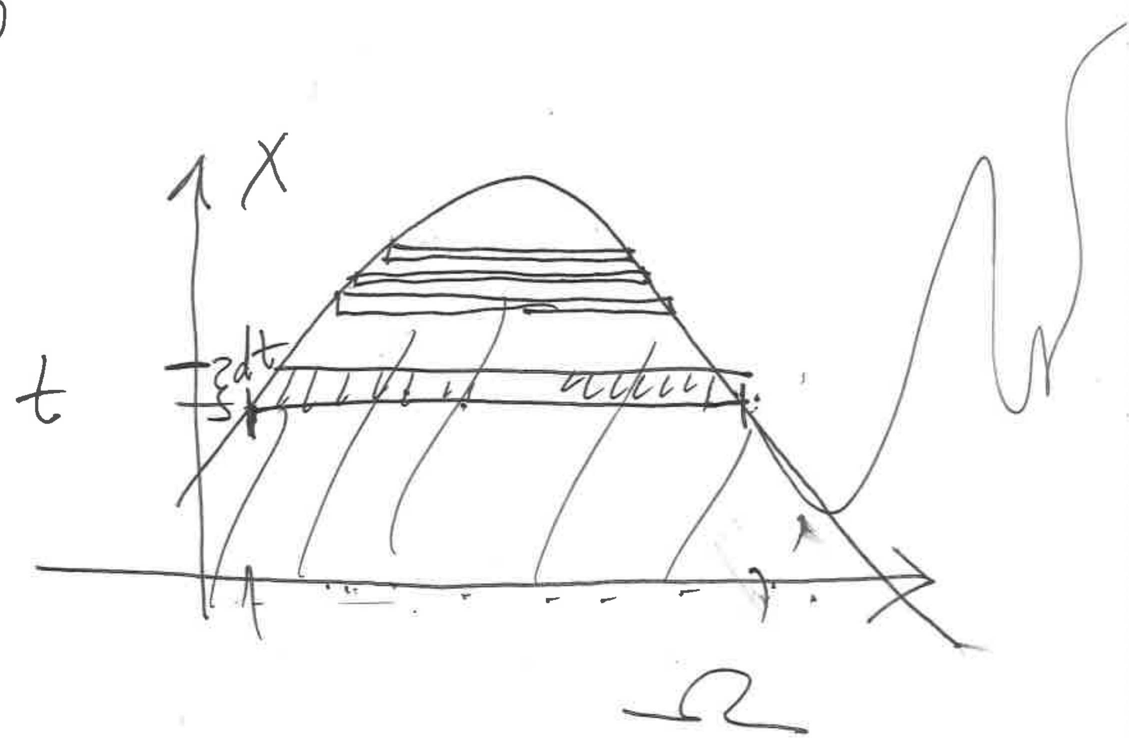
③ "Layer Cake Formula"

If $X \geq 0$ almost surely

then $EX = \int_0^\infty P(X \geq t)\, dt$

$\int_\Omega X \, dP$





More genlly: φ is a non-negative increasing diff fn,

$$E\,\varphi(X) = \int_0^\infty \varphi'(t)\, P(X \geq t)\, dt$$

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(4) ~~Unconscious~~ Lazy Statistician Formula.

Say X has PDF ϕ

& f is a non-random function.

$$EX = \int_{-\infty}^{\infty} x\, \phi(x)\, dx$$

$$E f(X) = \int_{-\infty}^{\infty} f(x)\, \phi(x)\, dx$$

① Independence
② Conditional Expectation.

Independence: ① A, B 2 events.

A is independent of B if $P(A \cap B) = P(A)P(B)$

$$\left(P(A \mid B) = P(A) \right).$$

② X & Y 2 RV's.

(Every event obtained from a question about X)
is independent from every event obtained from a question about Y.

Eg: $\{X < \alpha\}$ is ind of $\{Y > \beta\}$ for every α, β.

$\sigma(X) = \sigma$-algebra generated by X

$=$ All events that can be observed from X.

$\sigma(X) =$ smallest σ-algebra that contains the events $\{X \leq \alpha\}$ for every $\alpha \in \mathbb{R}$.

Can check $\sigma(X) \ni \{X \in (a, b)\}$

$\{e^X < \ln(5X)\}$.