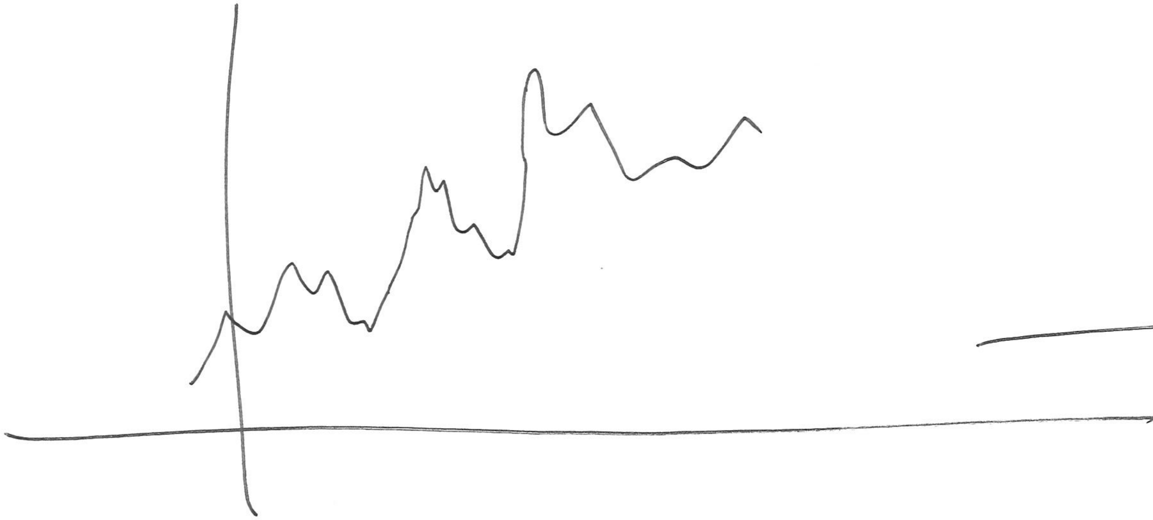
Midterm 1 : Thu Feb 7^{th} (In class). (30%)

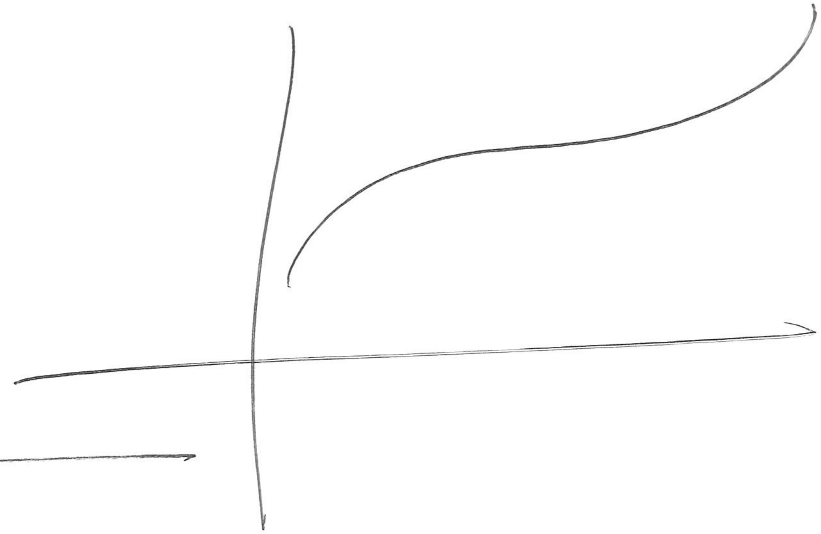
Final : Wed Mar 6^{th} ⟵ Time TBA. (60%).

Make up Final : Wed Mar 20^{th} (5:00 PM).

HW : Due Thursdays. (2:44:59.999). (10%).

TA: David Itkin.





ODE: $dS = \alpha S \, dt$ $\quad \left(\dfrac{dS}{dt} = \alpha S \right)$

Stocks: Geometric Brownian Motion.

(Brownian motion)

$$dS = \underbrace{\alpha S \, dt}_{\text{“mean return rate”}} + \sigma S \underbrace{dW}_{\text{“noisy flutions”}}$$

Use these models to Price securities.

① European Call:

$S \longrightarrow$ stock price.

European call: Maturity T, strike K.

Value of at time T: $(S-K)^+$

Q: What is the value of at time 0 (now)?

Black Scholes Formula:

$c(t, x)$ = Value of a E. call at time t

given the stock price is x.

B.S. $\cancel{c(x}$ $c(t,x) = x\, N\big(d_+(T-t,x)\big)$

$$- K e^{-r(T-t)} N\big(d_-(T-t,x)\big).$$

$N \longrightarrow$ CDF of Normal.

$r \longrightarrow$ interest rate.

$$d_{\pm} = \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\overset{\tau}{\cancel{(T-t)}}\right]$$

$d_{\pm}(x,\tau)$.

$d_{\pm}(\tau,x)$

$\sigma \longrightarrow$ volatility (% volatility).

Idea behind finding this formula:

Replicating Portfolio:

↳ Sell a call for \$ $c(t, x)$.

Invest c in → Money market (return rate r)

→ the stock S...

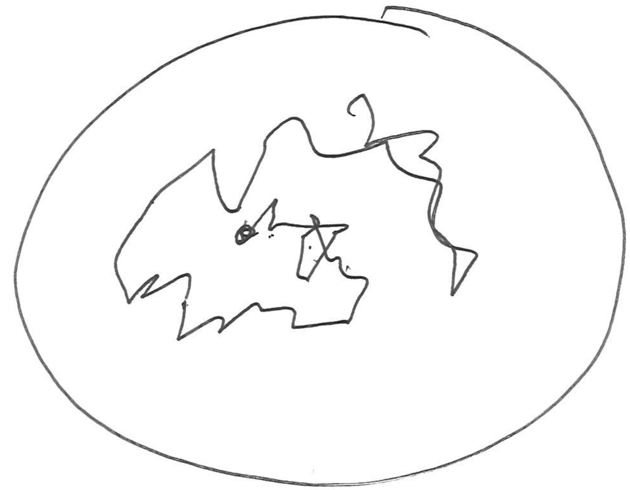
Goal: at time t, left with $(S-K)^+$

After B.S. → Risk Neutral Measures.

→ (~~FTAP~~) (American Options).

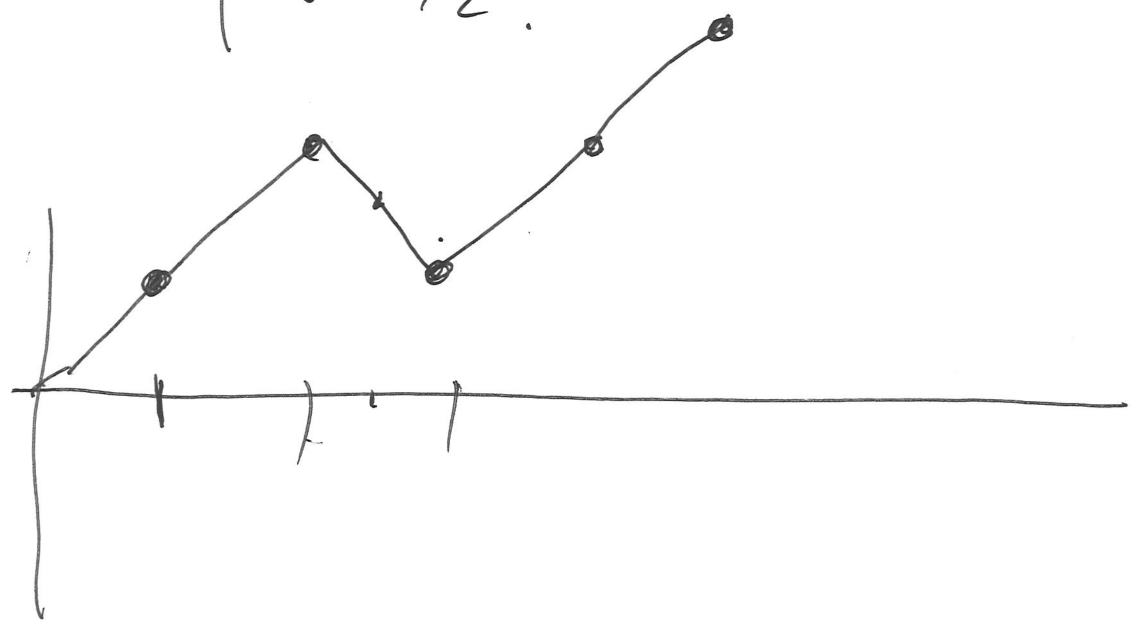
① Brownian Motion:

"Continuous time Random walk".



Let $\xi_1, \xi_2, \ldots$ be iid.

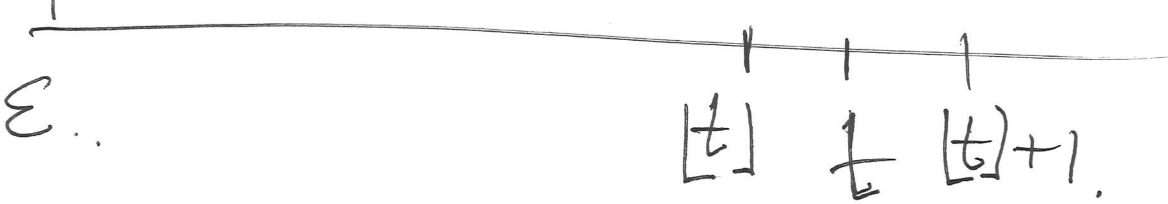
$$\xi_i = \begin{cases} +1 & \text{prob } 1/2 \\ -1 & \text{prob } 1/2. \end{cases}$$



$$S(n) = \sum_{i=1}^{n} \xi_i \qquad (n \in \mathbb{N}).$$

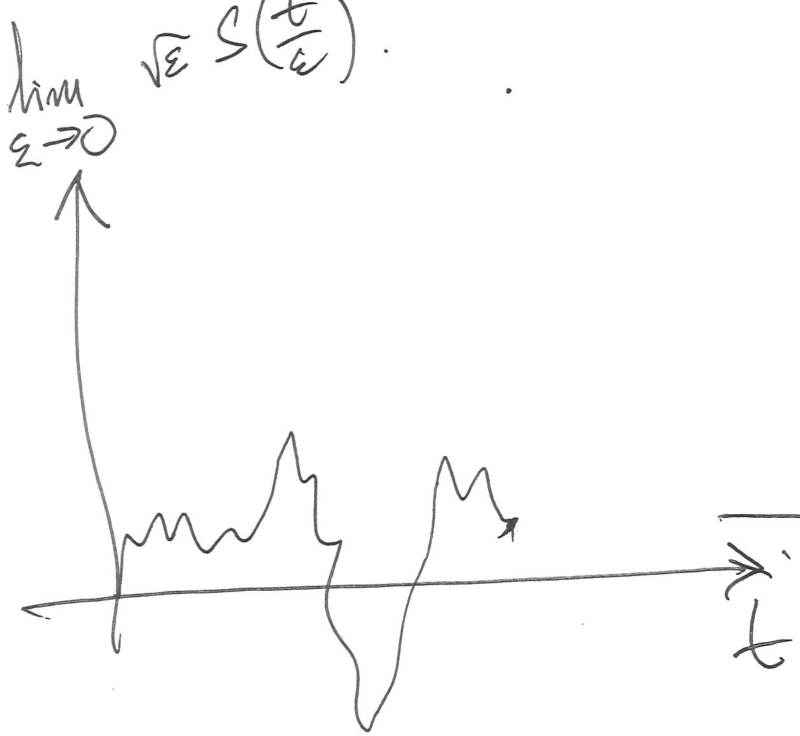
$$S(t) = S(\lfloor t \rfloor) + (t - \lfloor t \rfloor)\,\xi_{\lfloor t \rfloor + 1} \qquad \left(\begin{array}{l} t \geq 0 \\ t \in \mathbb{R} \end{array}\right).$$

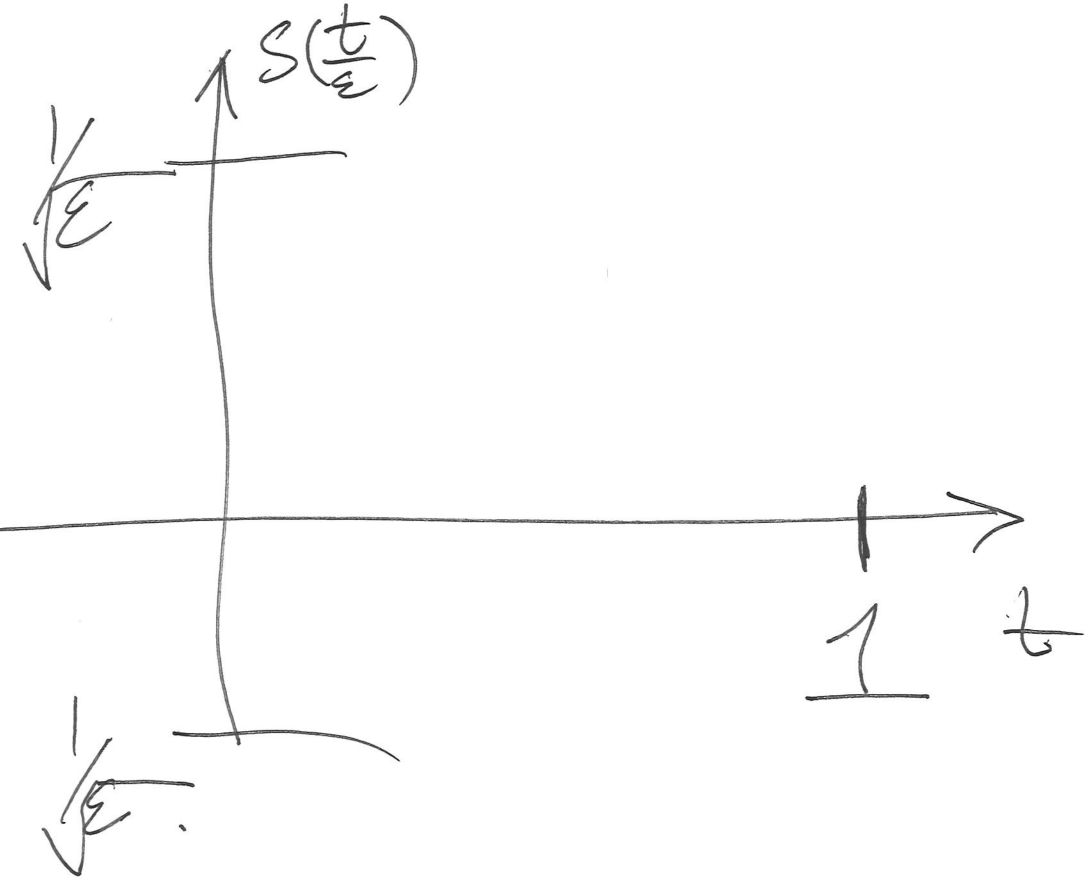
Coin flips at integer multiples of the step size ε.



$$S^{\varepsilon}(t) = \sqrt{\varepsilon}\, S\left(\frac{t}{\varepsilon}\right)$$

(Choose $\sqrt{\varepsilon}$ prefactor to ensure $\mathrm{Var}(S^{\varepsilon})$ remains bdd as $\varepsilon \to 0$)





Thm: $\lim_{\varepsilon \to 0} S^{\varepsilon}(t) = \lim_{\varepsilon\to 0} \sqrt{\varepsilon}\, S\left(\frac{t}{\varepsilon}\right)$. exists.

the limti is Brownian Motion.

Better (more useful & ~~less intuitive~~) char of B. M.

Definition 1 A Brownian Motion is a continuous process that has stationary independent increments.

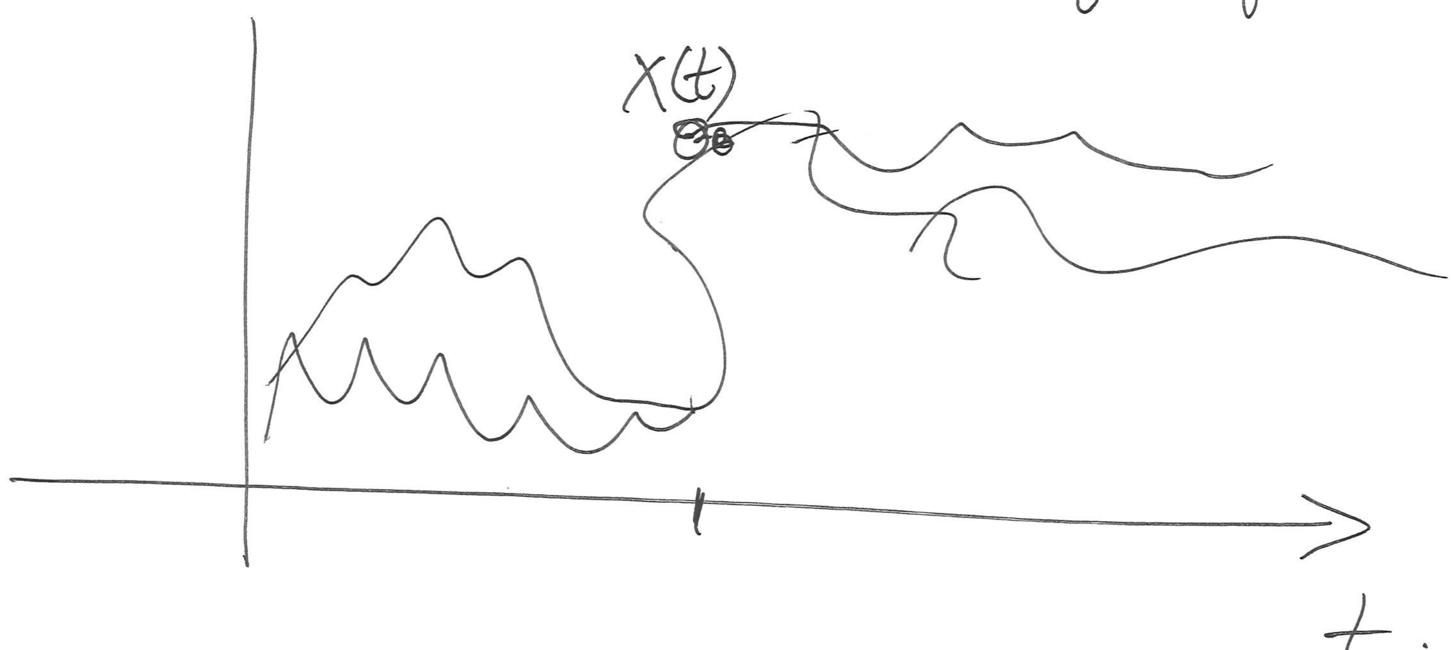
① Process: A colletion of random variables. $\{X(t) \mid t \geq 0\}$.

$t \rightarrow$ denotes time.

$X(t) \longrightarrow$ Random variable.

② Contianuous process:

(2a) ~~The~~ A trajectory (aka sample path) of the process X is the outcome of one realization of each of the RV's $X(t)$, viewed as a fn of t.



(2b) A cts process is a process whose trajectories are ALL continuous.

→ For every $t \geq 0$, $\boxed{\lim_{s \to t} X(s) = X(t)}$.

(3) A process has stationary increments if

For EVERY $h > 0$

the dist of $X(t+h) - X(t)$ does NOT depend on t

(it can depend on h).

$$S(n) = \sum_{i=1}^{n} \xi_i$$

$$S(n+1) - S(n) = \xi_{n+1} \leftarrow$$ dist is ind of n.

$$S(n+2) - S(n) = \xi_{n+2} + \xi_{n+1} \leftarrow$$

$$S^{\varepsilon} \rightarrow S^{\varepsilon}(t+h) - S^{\varepsilon}(t)$$

↳ stationary provided t is an int mult of ε

$\geq h$

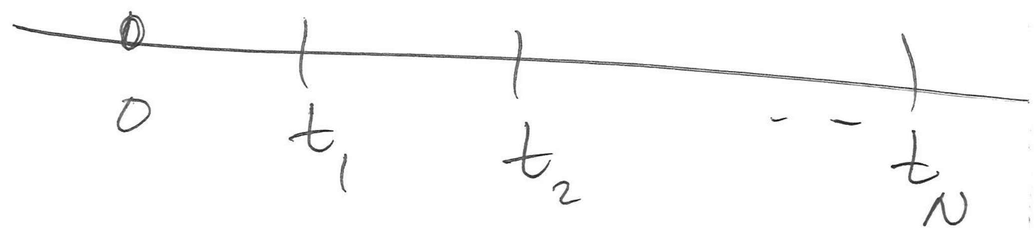
Otherwise there are error terms.

(Can check → these vanish as $\varepsilon \rightarrow 0$).

4 Independent Increments:

A process X is said to have independent increments if for any seq of times $0 \le t_0 < t_1 < t_2 \dots < t_N$

$X(t_0)$, $X(t_1) - X(t_0)$, $X(t_2) - X(t_1)$ ....



The RVs $X(t_0)$, $X(t_1) - X(t_0)$, ... $\cancel{X(t_N) - X(t}$, $X(t_N) - X(t_{N-1})$

are all JOINTLY independent.

$$S^{\varepsilon} = \sqrt{\varepsilon}\, S\left(\frac{t}{\varepsilon}\right)$$

Say s, t are integer multiples of ε.

$$S^{\varepsilon}(t) - S^{\varepsilon}(s) \sim \sqrt{\varepsilon} \sum_{i=1}^{(t-s)/\varepsilon} \xi_i \xrightarrow[\text{CLT}]{\varepsilon \to 0} N(0, t-s)$$

If s, t are not int multiples of ε.

get error terms. But as $\varepsilon \to 0$ they vanish!.

$\Rightarrow$ as $\varepsilon \to 0$ $\quad S^{\varepsilon}(t) - S^{\varepsilon}(s) \longrightarrow N(0, t-s).$

Definition 2: A Brownian Motion (BM) is a cts process W such that

1. W has independent increments.
2. For $s < t$, $W(t) - W(s) \sim N(0, \sigma^2 (t-s))$.

$(\sigma > 0$ is a fixed number$)$.

Rmk: When $\sigma = 1$, this is called a standard BM.

Claim: Def 1 $\Longleftrightarrow$ Def 2.

Review of Probability.

\mathcal G.

① Probability triple $(\Omega, \mathcal{G}, P)$.

$\Omega \longrightarrow$ sample space. (non-empty set).

$\mathcal{G} \longrightarrow \sigma$ - algebra (subsets of Ω).

(Non-empty collection of <u>events</u> of which the probability is known).

$P \longrightarrow$ Probability measure.

For any event $A \in \mathcal{G}$, $P(A)$ represents the prob of A happening.