2 IOU's FROM CLASS (proof of the div thm).

Lemma 1: $A \longrightarrow 3\times 3$ matrix. $u, v \in \mathbb{R}^3$. (A inv).

$$(Au) \times (Av) = (\operatorname{adj} A)^T (u \times v)$$

$\operatorname{adj} A$ = classical adjoint of A = transpose of co-factor matrix.

Recall: $C_{ij} = \det(M_{ij})(-1)^{i+j}$, $M_{i,j}$ = matrix A with i^{th} row & j^{th} col removed.

cofactor matrix $= \left(C_{i,j} \right)$. $\operatorname{adj}(A)$ = transpose of co-factor matrix.

$$\mathrm{adj}(A)\cdot A = \det(A)\cdot I_3 \quad \left(\Leftrightarrow A^{-1} = \frac{1}{\det(A)}\,\mathrm{adj}(A)\right).$$

Pf of lemma: (A inv).

$$(Au \times Av) = (\mathrm{adj}\, A)^T (u \times v)$$

$$\Leftrightarrow \forall w \in \mathbb{R}^3 \quad (Au \times Av)\cdot w = \left[(\mathrm{adj}\, A)^T (u\times v)\right]\cdot w$$

$$\Leftrightarrow (\because A \text{ is inv}) \quad (Au \times Av)\cdot(Aw) = \left[(\mathrm{adj}\, A)^T u\times v\right]\cdot Aw \quad \cdots \; \ast$$

Pf of $\ast$: ① RHS: $\left[(\mathrm{adj}\, A)^T u\times v\right]\cdot Aw = (u\times v)\cdot\left[(\mathrm{adj}\, A)\, Aw\right]$

$$= \det(A)\,(u\times v)\cdot w$$

② LHS: $(Au \times Av) \cdot Aw = \det \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ Au & Av & Aw \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$

$$= \det \left(A \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ u & v & w \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \right)$$

$$= (\det A) \det \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ u & v & w \\ \downarrow & \downarrow & \downarrow \end{pmatrix} = (\det A)(u \times v) \cdot w.$$

QED!

Lemma 2 from class: $F: U \to \mathbb{R}^3$, C^1. $g: (0,1)^3 \to U$ C^2 diffeo

$$\nabla \cdot \left(\operatorname{adj}(Dg)\, F \circ g \right) = (\nabla \cdot F) \circ g \, (\det Dg) \quad \cdots \quad (**)$$

Pf: Let $A = \operatorname{adj}(Dg)$. $A = (a_{ij})$. $\boxed{A\, Dg = (\det Dg)\, I}$

LHS of (**): $\nabla \cdot (A\, f \circ g) = \sum_i \partial_i (a_{ij}\, f_j \circ g)$.

$$= \underbrace{\sum_{ij} \partial_i a_{ij}\; f_j \circ g}_{\mathrm{I}} + \underbrace{\sum_{i,k,j} a_{ij} \left(\partial_k f_j \big|_g \; \partial_i g_k \right)}_{\mathrm{II}}.$$

I: Mixed partials are equal $\Rightarrow \sum_i \partial_i a_{ij} = 0$ (you check).

$$\mathrm{II}: \sum_{i,k} a_{ij}\, \partial_k f_j\Big|_g\, \partial_i g_k = \sum_{j,k}\Big(\sum_i \underbrace{\partial_i g_k}_{(Dg)_{ki}}\ \underbrace{a_{ij}}_{(A)_{ij}}\Big)\, \partial_k f_j\Big|_g$$

$$= \sum_{jk}\Big(\big(Dg\ A\big)_{k,j}\Big)\, \partial_k f_j\Big|_g$$

$$= \sum_{j,k} (\det Dg)\, \delta_{kj}\, \partial_k f_j\Big|_g \qquad \left(\begin{array}{l} \delta := \text{Kronecker } \delta \\ \delta_{kj} = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases} \end{array}\right).$$

$$= \sum_k (\det Dg)\, \partial_k f_k\Big|_g$$

$$= (\det Dg)\ (\nabla \cdot F) \circ g$$

QED.