GREENS THEOREM.

① Piecewise C^1. Def: A fn $f : [a,b] \longrightarrow \mathbb{R}^d$ is called piecewise C^1 if

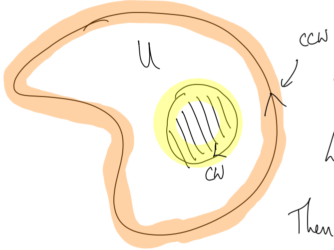
① $\exists F \subseteq [a,b]$ finite $\ni f : [a,b] - F \longrightarrow \mathbb{R}^d$ is C^1.

② $f : [a,b] \longrightarrow \mathbb{R}^d$ is cts.

& ③ $\forall x \in F$, $\lim_{y \to x^-} Df$ exists & $\lim_{y \to x^+} Df$ exists (need not be equal)

Typical example: $f(x) = |x|$. (piecewise $C^1 \approx C^1$, with finitely many corners) (no cusps, no jumps.)

Thm (Greens Theorem) ① $U \subseteq \mathbb{R}^2$ is $\boxed{\text{bdd}}$, ② ∂U = finite union of piecewise C^1 curves.



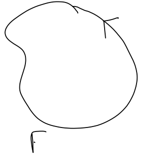
∂U = Orange curve + yellow curve.

Let $F : \overline{U} \to \mathbb{R}^2$ be C^1

Then: $$\underbrace{\oint_{\partial U} F \cdot dl}_{\text{line int}} = \underbrace{\int_U (\partial_1 F_2 - \partial_2 F_1)\, dA}_{\text{area int.}}$$

Note:

$$\int_\Gamma F \cdot dl.$$



$$\int_\Gamma F \cdot dl.$$

1. $\oint$ denotes a line integral along a closed curve.
2. When we perform line integrals along closed curves, we need to specify an orientation (or a direction of traversal).
3. Greens theorem: $\oint_{\partial U}$ $\longrightarrow$ Exterior boundary traversed counter clockwise.

& all interior boundaries are traversed clockwise.

Strategy of Proof:

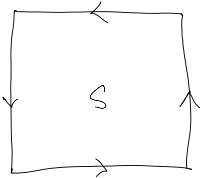
① Prove Green's thm on a square.

② Prove Green's thm assuming ∃ a C^2 coordinate change fn. between the unit square S & the domain U.

③ Divide U into finitely many pieces that are in the form in ②, & prove Green's thm.

① Pf of Greens thm for the unit square.

$S = [0,1] \times [0,1]$.



NTS. $\displaystyle\int_{\partial S} F \cdot d\ell = \int_S (\partial_1 F_2 - \partial_2 F_1)\, dA$.

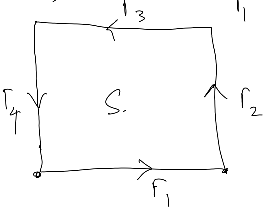
Note: $$\int_S (\partial_1 F_2 - \partial_2 F_1)\, dA = \int_{x_2=0}^{1} \underbrace{\int_{x_1=0}^{1} \partial_1 F_2 \, dx_1}_{\text{FTC}} dx_2 - \int_{x_1=0}^{1} \underbrace{\int_{x_2=0}^{1} \partial_2 F_1 \, dx_2}_{\text{FTC}} dx_1$$

$$= \int_{x_2=0}^{1} \Big(F_2(1, x_2) - F_2(0, x_2)\Big)\, dx_2 - \int_{x_1=0}^{1} \Big(F_1(x_1, 1) - F_1(x_1, 0)\Big)\, dx_1$$

$$(*) \quad = \underbrace{\int_{x_2=0}^{1} F_2(1, x_2)\, dx_2}_{\int_{\Gamma_2}} - \underbrace{\int_0^1 F_2(0, x_2)\, dx_2}_{\int_{\Gamma_4}} - \underbrace{\int_0^1 F_1(x_1, 1)\, dx_1}_{\int_{\Gamma_3}} + \underbrace{\int_0^1 F_1(x_1, 0)\, dx_1}_{\int_{\Gamma_1} F \cdot dl}.$$

Compute $\int_{\Gamma_1} F \cdot dl$: Let $\gamma(t) = (t, 0)$

$$\int_{\Gamma_1} F \cdot dl = \int_0^1 F(t, 0) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt$$



$$= \int_0^1 F_1(t,0)\, dt = \int_0^1 f_1(x_1, 0)\, dx_1$$

From (*) see $\displaystyle\iint_S (\partial_1 F_2 - \partial_2 F_1)\, dA = \sum_i \int_{\Gamma_i} F \cdot d\ell = \int_{\partial S} F \cdot d\ell.$

QED.
(on the sq).

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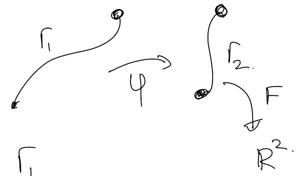
(2) Suppose now $U = \varphi(S)$, φ C^2 diffeo.

Lemma: (Change of variables for line integrals).

Say $\Gamma_1, \Gamma_2 \subseteq \mathbb{R}^2$ are two C^1 curves.

Say $F : \Gamma_2 \to \mathbb{R}^2$ is C^1 & $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ a C^1 diffeo such that $\varphi(\Gamma_1) = \Gamma_2$ & preserves orientation.

Then $\displaystyle\int_{\Gamma_2} F \cdot d\ell = \int_{\Gamma_1} (D\varphi)^T (F \circ \varphi) \cdot d\ell$



Proof: Let $\gamma : [0,1] \to \Gamma_1$ be a param of Γ_1

Then $\varphi \circ \gamma$ is a param of Γ_2 (in the given direction of traversal).

$$\Rightarrow \int_{\Gamma_2} F \cdot d\ell = \int_0^1 F \circ (\varphi \circ \gamma(t)) \cdot \left((\varphi \circ \gamma)'(t) \right) dt$$

$$= \int_0^1 F \circ \varphi \circ \gamma(t) \cdot \left(D\varphi_{\gamma(t)} \, \gamma'(t) \right) dt .$$

$$= \int_0^1 \left[\left(D\varphi_{\gamma(t)} \right)^T F \circ \varphi \circ \gamma(t) \right] \cdot \gamma'(t) \, dt \qquad (\because u \cdot Av = (A^T u) \cdot v).$$

$$= \int_{\Gamma_1} (D\varphi)^T (F \circ \varphi) \cdot d\ell \qquad \text{QED}.$$

② Part II of pf of Greens thm:

① Assume $U = \varphi(S)$, $S =$ unit square.
$\varphi =$ a C^2 diffeomorphism, orientation preserving.

Let $F : \overline{U} \longrightarrow \mathbb{R}^2$ be C^1. NTS $\displaystyle\int_U (\partial_1 F_2 - \partial_2 F_1) \, dA = \int_{\partial U} F \cdot d\ell$

Note : $\varphi(\partial S) = \partial U$. Assume φ is orientation preserving.

$$\Rightarrow \int_{\partial U} F \cdot dl = \int_{\varphi(\partial S)} F \cdot dl = \int_{\partial S} (D\varphi)^T F \circ \varphi \cdot dl \quad \cdots \quad (**)$$

Let $G = (D\varphi)^T F \circ \varphi$.

Greens thm on the sq $\Rightarrow \int_{\partial U} F \cdot dl = \int_{\partial S} G \cdot dl = \int_S (\partial_1 G_2 - \partial_2 G_1) \, dA$.

You check : $\partial_1 G_2 - \partial_2 G_1 = (\partial_1 F_2 - \partial_2 F_1) \circ \varphi \; \underbrace{\det(D\varphi)}_{>0}$ (orientation preserving).

$$\Rightarrow \int_{\partial U} F \cdot dl = \int_{S} (\partial_1 F_2 - \partial_2 F_1) \circ \varphi \ |\det(D\varphi)| \ dA$$

$$\underset{\text{coordinate change}}{=\!=} \int_{U} \left(\partial_1 F_2 - \partial_2 F_1\right) dA.$$

QED.