**Last time:** Inverse function thm:

If $f : U \longrightarrow \mathbb{R}^d$, $C^1$ ($U \subseteq \mathbb{R}^d$ is open).

$a \in U$, $Df_a$ is invertible.

Then $\exists\, U' \ni a$ (open) & $V \ni f(a)$ open such that

$f : U' \longrightarrow V$ is $C^1$, $\boxed{\text{bij}}$ & $f^{-1}$ is $C^1$.

Pf (Mostly done last time): 1 $a = 0$, $f(a) = 0$, $Df_a = I$.

2 Intuition: Close to $0$, $f(x) \approx x$.

Showed $\exists R > 0$ s.t. $\frac{1}{2}|x-y| \leq |f(x) - f(y)| \leq \frac{3}{2}|x-y|$

$\forall x, y \in B(0,R)$. [Idea: $F(x) = f(x) - x$].

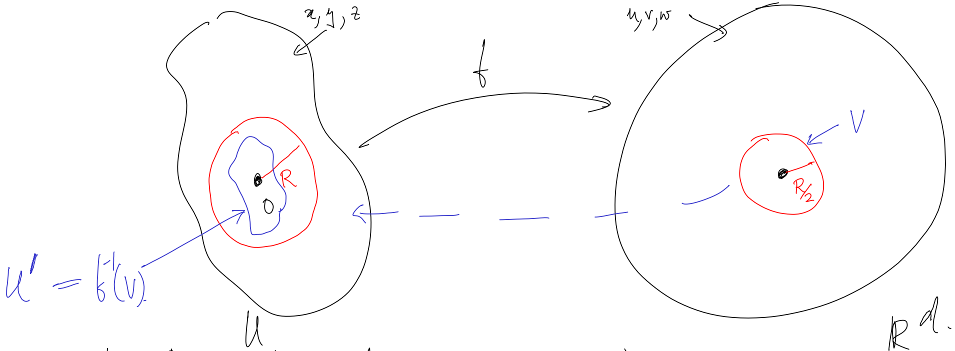
$\Rightarrow f$ is inj on $B(0,R)$

(3) Surj: Claim: $\forall u \in B(0, R/2)$, $\exists x \in B(0,R)$ s.t. $f(x) = u$.

[ $f(x) = u \iff f(x) - x + x = u \iff x = \underbrace{u - F(x)}_{G(x)}$.

C. map on $G$ to show $\exists x$ ].

(4) Let $U' = f^{-1}(B(0, R/2)) \cap B(0,R)$.




Note $U'$ is open! ($f$ cts & $V$ is open).

$f : U' \longrightarrow V$ is inj & surj $\Rightarrow$ $f : U' \longrightarrow V$ is bij.

$\Rightarrow$ Let $g = f^{-1}$, $g : V \longrightarrow U'$

(5) NTS: $g$ is $C^1$. First show $g$ is diff.

Know $f$ is diff.

Know $f(\underbrace{x+h}_{y}) = f(x) + Df_x\, h + e(h)$ $\quad (x, y \in U')$.

for some fn $e$ $\ni$ $\lim_{h \to 0} \frac{e(h)}{|h|} = 0$.

$$\Rightarrow \underbrace{f(y)}_{v} = \underbrace{f(x)}_{u} + Df_x\,(y-x) + e(y-x).$$

Let $u = f(x)$, $v = f(y)$. $\Leftrightarrow y = g(v)$ & $x = g(u)$

$$\Rightarrow v = u + Df_{g(u)}\left(g(v) - g(u)\right) + e\left(g(v) - g(u)\right).$$

$$\Leftrightarrow \quad g(v) = g(u) + \left(Df_{g(u)}\right)^{-1}(v-u) - \left(Df_{g(u)}\right)^{-1} e\big(g(v)-g(u)\big).$$

To show $g$ is diff: Guess $Dg_u = \left(Df_{g(u)}\right)^{-1}$ ← linear transformation.

To show this we only NTS. $\displaystyle\lim_{v\to u} \frac{\left(Df_{g(u)}\right)^{-1} e\big(g(v)-g(u)\big)}{|v-u|} = 0$

$\Leftrightarrow$ show $\displaystyle\lim_{v\to u} \frac{e\big(g(v)-g(u)\big)}{|v-u|} = 0$

<u>Pf</u>: Know $\displaystyle\lim_{h\to 0} \frac{e(h)}{|h|} = 0$

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0 \ni |h| < \delta \Rightarrow |e(h)| < \varepsilon |h|$

Say $|v - u| < \delta/2$.

$$|g(v) - g(u)| = |y - x| \leq 2 |f(y) - f(x)| = 2 |v - u| < \delta.$$

$$\Rightarrow |e(g(v) - g(u))| < \varepsilon |g(v) - g(u)| < 2\varepsilon |v - u|$$

$$\Rightarrow \frac{|e(g(v) - g(u))|}{|v - u|} < 2\varepsilon. \Rightarrow \lim_{v \to u} \frac{e(g(v) - g(u))}{|v - u|} = 0$$

$\Rightarrow g$ is diff!

⑥ $g$ is $C^1$. Pf: $Dg_u = \left(Df_{g(u)}\right)^{-1} \leftarrow$ formula

$\Rightarrow Dg_u$ is cts.

QED.

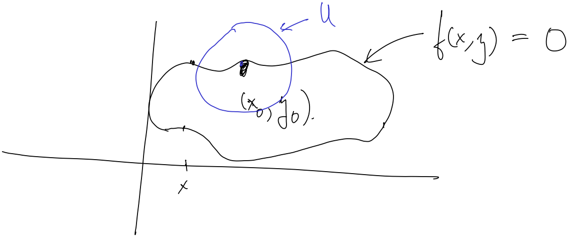
# IMPLICIT FUNCTION THEOREM.

Say $f : \mathbb{R}^2 \to \mathbb{R}$. ($C^1$). Consider the eq $f(x,y) = 0$.

Say we know $f(x_0, y_0) = 0$.

Want: write all solutions of $f(x,y) = 0$

in the form $\boxed{y = g(x)}$.




Eg: $ax + by = c$

Solve for $y$: $y = \dfrac{c - ax}{b}$

Works if $b \neq 0$.

Eq $f(x,y) = ax + by$.

$b \neq 0 \iff \partial_y f \neq 0$ & in this case can solve for $y$.

Illy $a \neq 0 \iff \partial_x f \neq 0$ " " " " $x$.

Theorem: (Implicit function theorem).

Suppose $f : \underbrace{\mathbb{R}^m}_{x} \times \underbrace{\mathbb{R}^n}_{y} \longrightarrow \mathbb{R}^n$ (Eg: $n = m = 1$).

$x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$.

Say for some $(x_0, y_0) \in \mathbb{R}^m \times \mathbb{R}^n$, $f(x_0, y_0) = c \in \mathbb{R}^n$

Note $$Df_{(x_0, y_0)} = \left( \overbrace{\boxed{\partial_x f_{(x_0, y_0)}}}^{m \quad \partial_x} \; \overbrace{\boxed{\partial_y f_{(x_0, y_0)}}}^{n \quad \partial_y} \right) \Big\} n$$

$\partial_x f_{(x_0, y_0)}$ = first $m$ cols of $Df_{(x_0, y_0)}$.

$\partial_y f_{(x_0, y_0)}$ = last $n$ cols " $Df_{(x_0, y_0)}$.

<u>If</u>: $\partial_y f_{(x_0, y_0)}$ is <u>invertible</u> then

$$\{(x,y) \mid f(x,y) = c\} \cap U = \{(x, g(x)) \mid x \in U_1\}$$

$\exists U \ni (x_0, y_0)$ open $(U \subseteq \mathbb{R}^m \times \mathbb{R}^n)$ & $\exists g : U_1 \to \mathbb{R}^n$ + $g(x_0) = y_0$

$\exists U_1 \ni x_0$ open $(U_1 \subseteq \mathbb{R}^m)$ & $g$ is $C^1$.

$ax + by = c$ & solve for $y$.

$y = \frac{c - ax}{b}$ $(b \neq 0)$.

Vector. $A \to$ matrix
$B \to$ matrix

$Ax + By = c$.
$\Leftrightarrow y = B^{-1}(c - Ax)$.

poss if $\boxed{B \text{ is inv.!}}$

$f(x,y) = Ax + By$

$Df = \left( \overbrace{A}^{\partial_x f} \quad \overbrace{B}^{\partial_y f} \right)$