① **GOAL:** Inverse function thm.

② **Quick Recall:** Contraction mapping principle.

① $C \subseteq \mathbb{R}^d$ closed. $f : C \longrightarrow C$ $\boxed{\text{cts}}$

② Suppose $\exists \lambda < 1$ s.t. $\forall x, y \in C$, $|f(x) - f(y)| \leq \lambda |x - y|$

Then $\Rightarrow \exists c \in C$ s.t. $f(c) = c$.

Pf: ① Pick any $x_0 \in C$. Set $x_1 = f(x_0), \dots x_{n+1} = f(x_n)$.

② Claim: (x_n) is convergent.

Pf: $|x_{n+1} - x_n| = |f(x_n) - f(x_{n-1})| \leq \lambda\, |x_n - x_{n-1}|$

$\Rightarrow |x_{n+1} - x_n| \leq \lambda^n |x_1 - x_0|$

$\Rightarrow \sum (x_{n+1} - x_n)$ is cgt $(\because \lambda < 1)$.

Note $x_{n+1} = x_0 + \sum_{k=0}^{n} (x_{k+1} - x_k)$

RHS cgt $\Rightarrow$ LHS cgt. QED.

③ Let $(x_n) \longrightarrow x$. Claim: $f(x) = x$. (note C closed $\Rightarrow x \in C$)

$f(x) \underset{(\text{cont})}{=} \lim f(x_n) = \lim x_{n+1} = x$

QED.

Thm (INVERSE FN THM) Say $U \subseteq \mathbb{R}^d$ open & $f : U \to \mathbb{R}^d$ is C^1.

Let $a \in U$ & suppose (Df_a) is invertible ($\Leftarrow$ $\det(Df_a) \neq 0$)

$\exists\, U' \ni a$ open & $V \ni f(a)$ (U, V open $\subseteq \mathbb{R}^d$) +

$f : U' \to V$ is C^1, invertible & $f^{-1} : V \to U'$ is C^1!

Remark: Suppose $f : U' \to V$ is C^1 & inv & $f^{-1} : U' \to V$ is also C^1.

Then $\forall\, a \in U'$, Df_a must be invertible!

Pf: Let $g = f^{-1}$. $g(f(x)) = \underset{\mathbb{R}^d}{\underset{\cap}{x}} \Rightarrow D(f \circ g)_a = I$

$\Rightarrow (Df)_{g(a)} \, Dg_a = I \Rightarrow Df_{g(a)}$, & Dg_a are both inv.

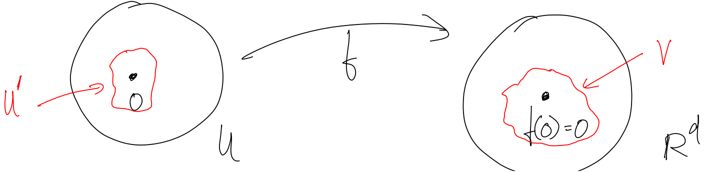
QED.

Proof of Inv fn thm:

① Assume $a = 0$, $f(a) = 0$ $(Df_a) = I$.

[If not, define $\tilde{f}(x) = (Df_a)^{-1}(f(x) - f(a))$].

& replace x with $x-a$ if necessary.



Want $f: U' \to V$ C^1, bij & f^{-1} is C^1.

<u>Intuition:</u> $f(x) = \underbrace{f(0)}_{0} + \underbrace{Df_0}_{I}\, x + \text{small}$

$= x + \boxed{\text{small.}}$ ← does not spoil invertibility.

(2) Let $F(x) = f(x) - x$.

<u>Claim:</u> $\exists R > 0$ such that $\forall x, y \in B(0,R)$, $|F(x) - F(y)| < \frac{1}{2}|x-y|$.

<u>Proof:</u> $DF_0 = Df_0 - I = 0$ (0 matrix).

$\Rightarrow \partial_i F_j(0) = 0$. $F \in C^1 \Rightarrow \forall \varepsilon > 0, \exists R > 0$ s.t. $|\partial_i F_j(x)| < \varepsilon$

whenever $|x| \leq R$.

$\Rightarrow \forall x \in B(0, R)$, $|\nabla F_j| < d\varepsilon$

Pick any $x, y \in B(0, R)$.

MVT: $\exists \xi$ on the line joig x & y $\ni F_i(x) - F_i(y) = (x-y) \cdot \nabla F_i(\xi)$

$\Rightarrow \left| F_i(x) - F_i(y) \right| \leq |x-y| \, |\nabla F_i(\xi)| \leq d\varepsilon \, |x-y|$.

Since this holds $\forall i \Rightarrow |F(x) - F(y)| \leq d \, (d\varepsilon |x-y|)$

$= d^2 \varepsilon |x-y|$.

Chose $\varepsilon = \dfrac{1}{2d^2} \Rightarrow |F(x) - F(y)| < \frac{1}{2} |x-y|$. QED (2).

③ $\forall x, y \in B(0,R)$ (some R from ②).

we have $\boxed{\frac{1}{2}|x-y| < |f(x)-f(y)| < \frac{3}{2}|x-y|}$

Pf: Know $|F(x)-F(y)| < \frac{1}{2}|x-y|$

$\Rightarrow |f(x)-x-(f(y)-y)| < \frac{1}{2}|x-y|$

$\Rightarrow |(f(x)-f(y))-(x-y)| < \frac{1}{2}|x-y|$

Δ ineq $\Rightarrow |f(x)-f(y)| < \frac{3}{2}|x-y|$

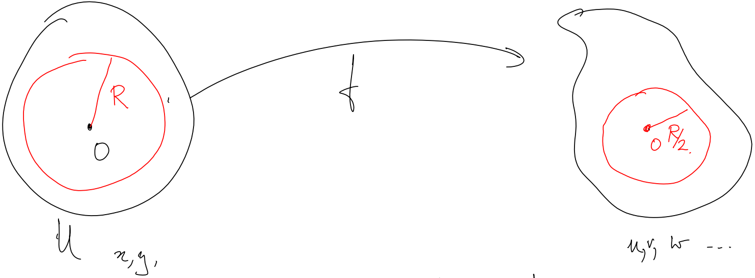
& $|f(x)-f(y)| > \frac{1}{2}|x-y|$.

QED ③.

④ $\Rightarrow f$ (restricted to $B(0,R)$) is injective!

Pf: $|f(x)-f(y)| > \frac{1}{2}|x-y|$. $\therefore x \neq y \Rightarrow f(x) \neq f(y)$! QED.

⑤ Surjectivity!



Claim: $\forall u \in B(0, R/2)$, $\exists x \in B(0,R)$ + $f(x) = u$.

Trick 1: Want $f(x) = u \iff f(x) - x + x = u$

$$\iff x = u - \underbrace{(f(x) - x)}_{F(x)} = \underbrace{u - F(x)}_{G}.$$

Let $G(x) = u - F(x)$. If $G(x) = x$, then $f(x) = u$.

Find fixed point of G using Contraction mapping.

Let $C = \overline{B(0, R)} = \{x \mid |x| \leq R\}$.

<u>Claim:</u> G is a contraction on C.

① $|G(x) - G(y)| = |F(x) - F(y)| \leq \frac{1}{2}|x - y|$. ✓

(2) $G: C \to C$. If $x \in C$, NTS $G(x) \in C$.

NTS $|G(x)| \leq R$..

$$|G(x)| \leq |u| + |F(x)| \leq \frac{R}{2} + |F(x) - F(0)|$$

$$\leq \frac{R}{2} + \frac{|x|}{2} \leq \frac{R}{2} + \frac{R}{2} = R. \checkmark$$

$\Rightarrow G$ maps C into C.

$\therefore$ C-mapping $\Rightarrow \exists\, x \in C \ni G(x) = x \Rightarrow f(x) = u$.

QED (Claim).

③ Altenite trick to show surj: minimize $|f(x) - u|^2$

---

⑥ Let $U' = f^{-1}\left(B(0, R/2)\right)$.

Above shows that $f : U' \to \underbrace{B(0, R/2)}_{V}$ is C^1 & bij.

$\Rightarrow f$ has an inverse on V.

IOU: ① f^{-1} is diff on V. (next time).