Continuity of Inverse Functions.

If $f : C \longrightarrow D$ is cts and bijective
$(C \subseteq \mathbb{R}^m)$ $(D \subseteq \mathbb{R}^n)$

Q: Must f^{-1} be cts.

Last time: Eg where f^{-1} is NOT cts!

Prove results that shows when f^{-1} IS cts.

(1) Say $I \subseteq \mathbb{R}$ (not $\mathbb{R}^d$) is an interval.

& $f : I \longrightarrow J$ is cts & bijective. ($J \subseteq \mathbb{R}$).

Thm : J is also an interval, f is (strictly) <u>monotone</u>

and $f^{-1} : J \longrightarrow I$ <u>is</u> cts!

(1d result, need domain I to be an interval)

Thm: If $C \subseteq \mathbb{R}^m$ is $\boxed{\text{sequentially cpt}}$, $D \subseteq \mathbb{R}^n$

If $f : C \to D$ is cts & bijective, then

$f^{-1} : D \to C$ must be cts!

Note: Know that f cts & C seq cpt $\Rightarrow$ D is also seq cpt.

Pf of Thm: Let $g = f^{-1}$.

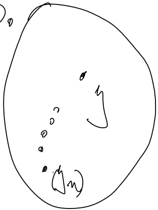
Will show g is cts by proving whenever $(y_n) \to y$ $(\in D)$

then $g(y_n) \longrightarrow g(y)$

Let (y_n) be any seq in D $\ni$ $(y_n) \to y \in D$.

NTS $\big(g(y_n)\big) \to g(y)$

Let $x_n = f^{-1}(y_n) = g(y_n)$ $\big(\Leftarrow f(x_n) = y_n \big)$.



D

Suppose $\big(g(y_n)\big) \not\to g(y)$.

i.e. Suppose $(x_n) \not\to x$ $\quad$ $\big($Let $x = g(y)\big)$.

By def, $\exists\, \varepsilon > 0 \ni \forall N \in \mathbb{N}, \exists\, n \geq N \ni |x_n - x| \geq \varepsilon$.

$\Rightarrow$ For $N = 1$, $\exists\, n_1 \geq 1$ + $|x_{n_1} - x| \geq \varepsilon$.

For $N = n_1$, $\exists\, n_2 \geq n_1$ + $|x_{n_2} - x| \geq \varepsilon$.

$\vdots$

$\exists\, n_k \geq n_{k-1}$ + $|x_{n_k} - x| \geq \varepsilon$.

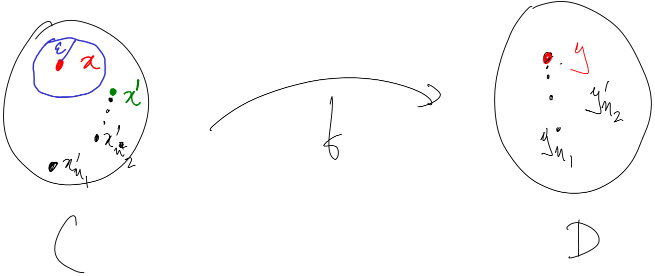
(x_{n_k}) is a seq in C (& C is seq cpt).

$\Rightarrow \exists$ a cgt subsequence $(x'_{n_k}) \longrightarrow x'$

Apply f: By continuity $(f(x'_{n_k})) \longrightarrow f(x')$

$\left(f(x'_{n_k})\right) = (y'_{n_k}) \longleftarrow$ a subsequence of (y_n)

$\Rightarrow (y'_{n_k})$ is convergent & $\boxed{(y'_{n_k}) \longrightarrow y.}$



f is cts $\Rightarrow \left(f(x'_{n_k}) \right) \longrightarrow f(x')$.

Also, $|x'_{n_k} - x| \geq \varepsilon \ \forall k \Rightarrow x' \neq x$.

$\Rightarrow (y_{n'_k}) \longrightarrow f(x')$

Already know $(y'_{n_k}) \longrightarrow y$

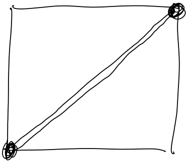
f is bijective.

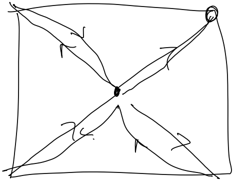
$$f(x) = y = \lim_{k\to\infty} y'_{n_k} = \lim_{k\to\infty} f(x'_{n_k}) = f(x') \neq f(x)$$

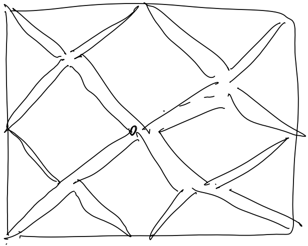
Contradiction QED.

Claim: (1) $\exists f : [0,1] \to [0,1]\times[0,1] \subseteq \mathbb{R}^2$ which is cts & surjetive.

(Goole "Space filling curve").







etc...

Q: Does there exist a cts Bijective fn from $[0,1] \longrightarrow [0,1] \times [0,1]$? (NO).

Q: Does $\exists$ a cts Bijective fn from $(0,1) \longrightarrow (0,1) \times (0,1)$? (NO)

Next Goal: (Friday). Uniform continuity.

Def: Let $U \subseteq \mathbb{R}^d$ be some set. & $f : U \to \mathbb{R}^n$ some fn.

① We say f is continuous on U if

$$\forall x \in U, \quad \boxed{\lim_{y \to x} f(y) = f(x).}$$

$$\Leftrightarrow \forall x \in U, \quad \forall \varepsilon > 0, \ \exists \delta > 0 \ni \ |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

② We say f is UNIFORMLY cts on U if

$$\forall \varepsilon > 0, \ \exists \delta > 0 \ni \ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

(Implicitly assume $x, y \in U$)

Difference between continuity & uniform continuity:

① For continuity the δ that "works" can depend on x.

② For uniform continuity, δ can NOT depend on x.

Clearly: ① If f is uniformly cts $\Rightarrow f$ is cts.

② Converse is false. (Eg: $U = \mathbb{R}$, $f(x) = x^2$

Claim f is cts but NOT U- cts).

(3) If $\boxed{K \text{ is seq cpt}}$ & $f: K \to \mathbb{R}^n$ is cts

then f is unifmly cts (on K).

If $\boxed{D \text{ is bdd}}$ & f is u-cts on $D \Rightarrow f$ is bdd.
