21-269 Vector Analysis: Final.

2019-05-07

- *This is a closed book test. No electronic devices may be used. You may not give or receive assistance.*
- *You have 3 hours. The exam has a total of 8 questions and* 80 *points.*
- *You may use any result proved in class or any regular homework problem* ***PROVIDED*** *it is* independent *of the problem you want to use the result in. (You must also* ***CLEARLY*** *state the result you are using.)*
- *The questions are roughly in the order the material was covered. I recommend looking through the whole exam before starting and doing questions you find easier first. My guess is that the difficulty of the questions is roughly Q6 $\approx$ Q7 $\approx$ Q1 $\leqslant$ Q2 $\approx$ Q5 $\leqslant$ Q4 $\approx$ Q3 $\leqslant$ Q8. Good luck* ◡̈.

$\boxed{10}$ 1. Is it possible to find a sequence (a_n) in $\mathbb{R}^3$ such that the sequence (a_n) is bounded, and $|a_n - a_m| \geqslant 1$ whenever $n \neq m$? If yes, find an example. If no, prove it.

$\boxed{10}$ 2. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f'(0) < 0$ and $f'(1) > 0$. Must there exist $c \in (0,1)$ such that $f'(c) = 0$? Prove it, or find a counter example.
Note: This is a special case of a homework problem. Please provide a complete solution here, and do not simply cite the homework problem in question. Also note that you are **NOT** told f is C^1, and so f' need not be continuous.

$\boxed{10}$ 3. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function. Show that $\partial_1\partial_2 f(0) = \partial_2\partial_1 f(0)$.
This is a special case of Clairaut's theorem and was proved in class. Please provide a complete proof here without simply citing the theorem. You may, however, use without proof any other result from class/homework (e.g. mean value theorem) that is independent of this result.

$\boxed{2}$ 4. (a) Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be C^1, and let $G = \{(x, f(x)) \mid x \in \mathbb{R}^2\} \subseteq \mathbb{R}^3$ be the graph of f. Given $a = (a_1, a_2, a_3) \in G$, find a basis of tangent space of G at the point a. Express each vector of the basis you find only in terms of a, $f(a)$ and/or partial derivatives of f at a. (No proof or justification required.)

$\boxed{8}$ (b) Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be a C^1 function, and let $M = \{f = 0\}$. Suppose $\nabla f(x) \neq 0$ for all $x \in M$. Given $a \in M$ such that $\partial_3 f(a) \neq 0$, show that $TM_a = \ker Df_a$.
Note: This is a special case of a result that was proved in class. However, please provide a complete proof here without simply citing the result from class. You may, however, assume without proof the result from the first part of this question, and other theorems we proved in class whose proof does not rely on this question.

$\boxed{10}$ 5. Find the constrained maximum of the function $f(x,y,z) = 2xy + yz$ on the unit sphere $x^2 + y^2 + z^2 = 1$.

$\boxed{10}$ 6. Given $R \in (0,1)$ define $U = \{x \in \mathbb{R}^2 \mid R < |x| < 1\}$. Compute $\displaystyle\int_U \frac{1}{(x_1^2 + x_2^2)^{3/2}} \, dA$.

$\boxed{10}$ 7. Let $U \subseteq \mathbb{R}^3$ be a bounded domain such that ∂U is a C^1 surface. Let $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1 function, $\hat{n}$ denote the outward pointing unit normal on ∂U, and $e_1 = (1,0,0)$ be the first standard basis vector. Find a function $G\colon \bar{U} \to \mathbb{R}$ such that

$$\int_{\partial U} (F \times \hat{n}) \cdot e_1 \, dS = \int_U G \, dV \,.$$

You should also find a formula expressing G in terms of F and/or partial derivatives of F, without involving integrals (e.g. $G(x) = 2F_1(x) + 6\partial_2 F_3(x) - 9\partial_3 F_1(x)$ is OK if correct, but $G(x) = \frac{1}{\operatorname{vol} U} \int_{\partial U} (F \times \hat{n}) \cdot e_1 \, dS$ is not).

$\boxed{10}$ 8. Let $U \subseteq \mathbb{R}^2$ be a bounded domain such that $0 \in U$ and ∂U is a closed C^1 curve. Compute

$$\oint_{\partial U} F \cdot d\ell \,, \qquad \text{where } F\colon \mathbb{R}^2 - \{0\} \to \mathbb{R}^2 \text{ is defined by} \quad F(x) = \frac{1}{2\pi |x|^2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} .$$

When computing the above line integral, we assume ∂U is traversed counter clockwise. You may **not** use anything about winding number in your solution.