$M \to$ mg (cts)
$M^2 \to$ Mg (cts)
$M(0) = 0$

} Show $M(t) = 0$ for all $t \geq 0$.

Know $M^2 - [M,M]$ is a mg

Know $M^2 - 0$ is a mg

Know if A is a cts adapted inc process with $A(0) = 0$
& $M^2 - A$ is mg $\Rightarrow A = [M,M]$.

$\Rightarrow 0 = [M,M]$. $\Rightarrow M$ has finite first variation.

$\Rightarrow M$ is a mg with finite 1st var $\Rightarrow M = 0$

Faster/Better way: $M^2 \to$ mg $\Rightarrow E M^2(t) = E M^2(0) = 0$

$M(t)^2 \geq 0$ & $E M(t)^2 = 0 \Rightarrow M = 0$ a.s.

Claim: M a mg, cts, $M(0) = 0$.

If M has finite 1st var $\Longrightarrow M(t) = 0$ for all t.

Pf: ① M has finite 1st var $\Longrightarrow [M,M] = 0$

Know $M^2 - [M,M]$ is a mg & $M(0)^2 - [M,M](0) = 0$

$\Longrightarrow E M(t)^2 = E [M,M](t) = 0 \Rightarrow M(t)^2 = 0$ a.s.

QED.

4.7: $\theta \in \mathbb{R}$ $\quad Z(t) = e^{\theta W(t) - \frac{\theta^2}{2} t}$

Given $s < t$, & a fn f, find a fn g ∋

$$E\left(f(Z(t)) \mid \mathcal{F}_s \right) = g(Z(s))$$

Sol: $E\left(f(Z(t)) \mid \mathcal{F}_s \right) = E\left(f\left(e^{\theta W(t) - \frac{\theta^2}{2} t} \right) \mid \mathcal{F}_s \right)$

$$= E\Big(f\Big(\underbrace{e^{\theta W(s) - \frac{\theta^2}{2} s}}_{Z_s} \cdot e^{\theta (W(t) - W(s)) - \frac{\theta^2}{2}(t-s)} \Big) \Big| \mathcal{F}_s \Big).$$

$$= E\left(f\left(Z(s) \, e^{\theta (W(t) - W(s)) - \frac{\theta^2}{2}(t-s)} \right) \Big| \mathcal{F}_s \right).$$

Independence Lemma

$$= \int_{y\in\mathbb{R}} f\left(Z(s)\, e^{\theta y\sqrt{t-s} - \frac{\theta^2}{2}(t-s)}\right) \frac{e^{-y^2/2}}{\sqrt{2\pi}}\, dy.$$

$$= \int_{-\infty}^{\infty} f\left(Z(s)\, e^{\theta y - \frac{\theta^2}{2}(t-s)}\right) \frac{e^{-y^2/2(t-s)}}{\sqrt{2\pi(t-s)}}\, dy.$$

Let $$g(x) = \int_{-\infty}^{\infty} f\left(x\, e^{\theta y - \frac{\theta^2}{2}(t-s)}\right) \cdot \frac{e^{-y^2/2(t-s)}}{\sqrt{2\pi(t-s)}}\, dy.$$

$$\Rightarrow E\left(f(Z(t)) \,\middle|\, \mathcal{F}_s\right) = g(Z(s)).$$

Q: Compute $E\left|\int_0^t W(s)\,ds\right|^{1/2}$

Sol: Itô won't work. ~~$E f(W(t))$~~

$$E\left(\int_0^t \sigma(s)\,dW(s)\right)^2 = E\int_0^t \sigma(s)^2\,ds. \quad \text{(Ito Isom)}.$$

Won't help here!

Note $\int_0^t W(s)\,ds$ is normally distributed.

① $\int_0^t f(s)\,W(s)\,ds$ (f not random) is normally dist.

& ② $\int_0^t \sigma(s)\,dW(s)$ (σ not random) ↗

~~Compute~~ Let $X(t) = \int_0^t W(s)\, ds$.

Know ~~..~~ $X \sim N(0, \sigma)$.

Compute σ^2: $\sigma^2 = E\, X(t)^2 = E\left(\int_0^t W(s)\, ds\right)\left(\int_0^t W(r)\, dr\right)$

$$= E \int_0^t \int_0^t W(s) W(r)\, dr\, ds = \int_0^t \int_0^t (s \wedge r)\, dr\, ds.$$

& compute. & find σ.

Suffices: $\int_{s=0}^t \int_{r=0}^s (\underbrace{r \wedge s}_{r})\, dr\, ds + \int_{s=0}^t \int_{r=s}^t (\underbrace{r \wedge s}_{s})\, dr\, ds$.

$$= \int_{s=0}^t \frac{s^2}{2}\, ds + \int_0^t (t-s) s\, ds = \underline{\qquad\qquad}$$

Find σ from that.

Now compute ~~E|X|~~ $E|X(t)|^{1/2}$.

~~At~~ Know $X(t) \sim N(0, \sigma^2)$

$$\Rightarrow E|X(t)|^{1/2} = \int_{-\infty}^{\infty} |y|^{1/2} e^{-y^2/2\sigma^2} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \, dy$$

$$E\left(\int_s^t e^{\lambda M(r)} W(r)\, dW(r) \,\Big|\, \mathcal{F}_s \right) \overset{\text{claim}}{=\!=} 0.$$

$N(t) = \int_s^t e^{\lambda M(r)} W(r)\, dW(r)$. N is a mg

$$\Rightarrow E\left(N(t) \,|\, \mathcal{F}_s\right) = N(s) = \int_s^s (\;) = 0.$$

4.9: $W \to$ BM.

Q: Is there an equivalent measure $\tilde{P}$ such that $t W(t)$ is a B.M.?

Know (Girsanov): there exists an equiv measure for which $t + W(t)$ is a BM.

Doesn't work for $tW(t)$!

Obs: Changing the measure does NOT change Q.V.

Reason $[X,X](T) = \lim \sum (X_{t_{i+1}} - X_{t_i})^2$ a.s. P

$\Rightarrow$ " " a.s. $\tilde{P}$.

<u>Obs 2</u>: QV of X is not t.

Compute $dX = t\,dW + W\,dt + 0$

$$\Rightarrow [X,X](t) = \int_0^t s^2\,ds = \frac{t^3}{3} \neq t.$$

$\Rightarrow$ Under any equiv measure $[X,X] = \frac{t^3}{3} \neq t$

$\Rightarrow$ X can not be a BM. (under any equiv measure).

<u>Q</u>: Is there an equiv measure $\tilde{P}$ $\ni$ X is a mg under $\tilde{P}$.

<u>Sol</u>: $dX = t\,dW + W\,dt = t\left(\underbrace{\frac{W(t)}{t}\,dt + dW}\right)$.

Try: Let $b(t) = \frac{W(t)}{t}$, $\tilde{W}(t) = \int_0^t b(s)\,ds + W(t)$.

Let $$Z(t) = \exp\left(-\int_0^t \frac{W(s)}{s}\, dW(s) - \frac{1}{2}\int_0^t \frac{W(s)^2}{s^2}\, ds \right).$$

$$d\tilde{P}_T = Z(T)\, dP$$

If Z is a mg $\Rightarrow \tilde{W}$ is a BM under $\tilde{P}$.

$\Rightarrow dX = t\, d\tilde{W} \Rightarrow X$ is a mg under $\tilde{P}$.

→ Know $dZ = -\sigma Z\, dW$

To check Z is a mg, need $$E\int_0^t \sigma(s)^2 Z(s)^2\, ds < \infty$$

Q: European put. strike K, mat T.

Q: R- portfolio is long on cash.

$p(t,x)$ = price of put.

Put call parity: ~~$p(t,x) - c(t,x) = -$~~

$$c(t,x) - p(t,x) = x - Ke^{-r(T-t)}$$

$$\Rightarrow p(t,x) = c(t,x) - x + Ke^{-r(T-t)}$$

R- Pf: $X(t)$ value at time t.

$\Delta(t)$ shares of stock

$\Gamma(t)$ cash.

$$dX(t) = \Delta(t)\, dS + \Gamma(t) \cdot r\, dt$$

Know $\Delta(t) = \partial_x f(t, S(t))$ (Delta Hedging).

$\Rightarrow$ Cash value $= \Gamma(t) = X(t) - \Delta(t) S(t)$.

$$= f(t, S(t)) - \partial_x f(t, S(t)) \cdot S(t).$$

Use formulae for Greeks from Black Scholes.

& check if $\Gamma(t) \geq 0$ or not.

Q: $X \sim N(0,1)$.

$Y = X + \alpha$ $\quad (\alpha \in \mathbb{R})$.

Find $\tilde{P}$ such that $Y \sim N(0,1)$ under $\tilde{P}$.

Sol: Write $d\tilde{P} = Z\, dP$ (Need to find Z).

Expect $Z = f(X)$ for some fn f.

Let $\varphi(x)$ = density of X = $\dfrac{e^{-x^2/2}}{\sqrt{2\pi}}$.

Let $\tilde{\varphi}(y)$ = density of Y under $\tilde{P}$ $\underset{=}{\text{Want}}$ $\dfrac{e^{-y^2/2}}{\sqrt{2\pi}}$.

Note: $\tilde{E}\, g(Y) = \int_{-\infty}^{\infty} g(y)\, \tilde{p}(y)\, dy$ (for any fn g).

Know $\tilde{E}\, g(Y) = E\, g(X+\alpha)\, f(X) = \int_{-\infty}^{\infty} g(x+\alpha) f(x)\, p(x)\, dx$

$(y = x+\alpha)$ $= \int_{-\infty}^{\infty} g(y)\, f(y-\alpha)\, p(y-\alpha)\, dy.$

$\Rightarrow \int_{-\infty}^{\infty} g(y)\, \tilde{p}(y)\, dy = \int_{-\infty}^{\infty} g(y)\, f(y-\alpha)\, p(y-\alpha)\, dy.$

for any fn g.

$$\Rightarrow \tilde{p}(y) = f(y-\alpha)\, p(y-\alpha).$$

Want $Y \sim N(0,1)$:

$$\Rightarrow \tilde{\phi}(y) = f(y-\alpha)\,\phi(y-\alpha) \Rightarrow \frac{\tilde{\phi}(y+\alpha)}{\phi(y)} = f(y)$$

$$\Rightarrow f(y) = \frac{e^{-\frac{(y+\alpha)^2}{2}}}{e^{-y^2/2}} = e^{-\frac{\alpha^2}{2} - \alpha y}$$

GBM: $dS = \alpha S\,dt + \sigma S\,dW$

Formula: $S(t) = S(0) \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right).$

$$S(s) = S(0) \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right)s + \sigma W(s)\right).$$

$$\Rightarrow \frac{S(t)}{S(s)} = \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right)(t-s) + \sigma\left(W(t) - W(s)\right) \right).$$

$$\Rightarrow S(t) = S(s) \cdot \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right)(t-s) + \sigma\left(W(t) - W(s)\right) \right).$$

(Q) **Security:** Pays $\left(S(T)^2 - K\right)^+$ at mat.

Price

Trick: $c(t, x; r, \sigma)$ = Black Scholes Formula.

$$c(t, S(t); r, \sigma) = \tilde{E}\left((S(T) - K)^+ \cdot e^{-r(T-t)} \middle| \mathcal{F}_t \right).$$

$S = GBM(r, \sigma)$ under $\tilde{P}$.

Compute $\tilde{E}\left(e^{-r(T-t)} \left(S(T)^2 - K\right)^+ \middle| \mathcal{F}_t \right)$

$S(t)^2 = GMB(2r + \sigma^2, 2\sigma)$

$$= e^{\left(-r + (2r+\sigma^2)\right)(T-t)} \underbrace{\widetilde{E}\left(e^{-(2r+\sigma^2)(T-t)} \left(S(T)^2 - K\right)^+ \,\middle|\, \mathcal{F}_t \right)}_{B.S.\left(2r+\sigma^2,\ 2\sigma\right)}.$$