Course Evalutions.

1. Ask Nicely $\longrightarrow$ 6 responses.
2. Incentive. $\longrightarrow$ 40% responses.
3. Theaten $\longrightarrow$

   Difficulty of Q1 on final $\approx \underline{\underline{1000}}\left(80 - \%\,\text{responses}\right)^{+} + \varepsilon$
4. Stop asking if 80% Response.

Recitation this week: NONE!

Review Session / Questions: $\longrightarrow$ 10:00 – 11:30 Cooper. Friday

Q1: $f \to$ $\boxed{\text{not random.}}$ $X(t) = \int_0^t f(s) W(s)\, ds$

Find dist of X.

Sol: ① X is normally dist.

$$X(t) = \lim_{\|P\| \to 0} \underbrace{\sum \underbrace{f(t_i)}_{\text{not random}} \underbrace{\cancel{(W_{t_{i+1}} - W_{t_i})}}_{\text{~~ind~~ ~~normals~~}}}_{\Rightarrow \text{ Normal.}} \Rightarrow X(t) \text{ is normal.}$$

$\underbrace{W(t_i)}\,(t_{i+1} - t_i)$

$(W_{t_0}, W_{t_1}, \dots W_{t_n})$ is Jointly Normal.

Compute $\mu = E\,X(t)$ & $\sigma^2 = E\,(X(t) - \mu)^2$

① $\mu = E \int_0^t f(s) W(s)\, ds = \int_0^t E\, f(s) W(s)\, ds = 0$

(2) $\sigma^2 = E\, X(t)^2$

Option 1: $E\, X(t)^2 = E\left(\int_0^t f(s)\, W(s)\, ds\right)^2$

$$= E\left(\int_0^t f(s)\, W(s)\, ds\right)\left(\int_0^t f(r)\, W(r)\, dr\right)$$

$$= E \int_{s=0}^{t}\int_{r=0}^{t} f(s) f(r)\, W(s)\, W(r)\, dr\, ds.$$

$$= \int_0^t \int_0^t f(s) f(r)\, (s \wedge r)\, dr\, ds.$$

Option 2: Choose $F(t) = \int_0^t f(s)\,ds$.

$$d(FW) = W F'\,dt + F\,dW + \underbrace{d[F,W]}_{\to 0}$$

$$= W f\,dt + F\,dW$$

$$\Rightarrow F(t)W(t) - 0 = \underbrace{\int_0^t W(s) f(s)\,ds}_{X(t)} + \int_0^t F(s)\,dW(s)$$

$$\Rightarrow X(t)^2 = \left(F(t)W(t) - \int_0^t F(s)\,dW(s)\right)^2$$

$$= F(t)^2 W(t)^2 + \left(\int_0^t F(s)\,dW(s)\right)^2 - 2F(t)W(t)\int_0^t F(s)\,dW(s)$$

$$\Rightarrow E X(t)^2 = t F(t)^2 + \int_0^t F(s)^2 \, ds - 2 F(t) E\left(W(t) \int_0^t F(s)\, dW(s)\right).$$

Compute last term: Let $M(t) = \int_0^t F(s)\, dW(s)$.

$\cancel{d\left(W \int_0^t F(s)\, dW(s)\right)}$

$$d\big(W(t) M(t)\big) = W\, dM + M\, dW + d[W, M].$$

$$= W\, dM + M\, dW + F(t)\, dt$$

$$\Rightarrow E\, W(t) M(t) - 0 = E \int$$

$$= 0 + 0 + \int_0^t F(s)\, ds.$$

$$\Rightarrow E X(t)^2 = t F(t)^2 + \int_0^t F(s)^2 \, ds - 2 F(t) \int_0^t F(s)\, ds.$$

Know $\int W \, dM$ is a mg.

$\int \underline{W} \, d\underline{M}$ need not be noml

Q2: $\sigma, \tau, \rho \longrightarrow$ not random (fns of t).

$M, N \rightarrow$ cts mg, $M(0) = N(0) = 0$.

Given $d[M,M](t) = \sigma(t) \, dt$

$d[N,N](t) = \tau(t) \, dt$

& $d[M,N](t) = \rho(t) \, dt$.

① Compute MGF

② If $\sigma = 1, \tau = 1, \rho = 0 \implies (M,N)$ is a 2D B.M.

Sol: Let $\lambda, \mu \in \mathbb{R}$,

$$f(t) = \varphi_{\lambda,\mu}(M(t), N(t)) = E\, e^{\lambda M(t) + \mu N(t)}$$

Itô: Let $Y(t) = e^{\lambda M(t) + \mu N(t)} = g(M, N)$

where $g(x, y) = e^{\lambda x + \mu y}$.

$$\partial_x g = \lambda g, \quad \partial_y g = \mu g, \quad \partial_x^2 g = \lambda^2 g, \quad \partial_y^2 g = \mu^2 g$$

$$\partial_{xy} g = \lambda \mu g.$$

$$\text{Itô} \Rightarrow dY = \lambda Y\, dM + \mu Y\, dN + \frac{1}{2}\Big(\lambda^2 Y\, d[M,M] + \mu^2 Y\, d[N,N]$$

$$2 \cancel{\partial_x} \lambda \mu Y\, d[M,N]\Big).$$

$$\Rightarrow Y(t) - \underbrace{Y(0)}_{1} = \int_0^t \lambda Y \, dM + \int_0^t \mu Y \, dN$$

$$+ \frac{1}{2} \int_0^t \left(\lambda^2 Y \sigma + \mu^2 Y \tau + 2 \lambda \mu Y \rho\right) ds.$$

$$\Rightarrow E\,Y(t) - 1 = f(t) - 1 = 0 + 0 + \frac{1}{2} \int_0^t \left(\lambda^2 \sigma + \mu^2 \tau + 2 \lambda \mu \rho\right) \underbrace{E\,Y(s)}_{f(s)} ds.$$

$$\Rightarrow f(t) = 1 + \frac{1}{2} \int_0^t \left(\lambda^2 \sigma(s) + \mu^2 \tau(s) + 2 \lambda \mu \rho(s)\right) f(s) \, ds.$$

$$\Rightarrow \frac{df}{dt} = \frac{\left(\lambda^2 \sigma + \mu^2 \tau + 2 \lambda \mu \rho\right)}{2} f$$

MGF

$$\Rightarrow f(t) = \underbrace{f(0)}_{1} \exp\left(\frac{1}{2} \int_0^t \left(\lambda^2 \sigma(s) + \mu^2 \tau(s) + 2 \lambda \mu \rho(s)\right) ds\right).$$

② Levy: $\sigma = 1$, $\tau = 1$, $\rho = 0$.

$$\Rightarrow \varphi_{\lambda,\mu}\big(M(t), N(t)\big) = E\, e^{\lambda M(t) + \mu N(t)}$$

$$= \exp\left(\frac{1}{2}(\lambda^2 + \mu^2)\, t\right).$$

$=$ MGF of 2D normal mean 0 & variance t.

$\Rightarrow \big(M(t), N(t)\big) \to$ Jointly normal $N\left(0, \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\right)$.

like 2D BM.

To show $\big(M(t), N(t)\big)$ is a 2D BM

NTS $M(t) - M(s)$ is ind & $\mathcal{F}_s$ (& same for N).

Same trick (before Midterm) : Compute. $E\left(e^{\lambda(M(t)-M(s))} \mid \mathcal{F}_s\right) \dots$

Compute $E\left(e^{\lambda(M(t)-M(s))} \,\middle|\, \mathcal{F}_s\right) = e^{\frac{\lambda^2}{2}(t-s)}$

$\left(\Rightarrow M(t)-M(s) \text{ is ind of } \mathcal{F}_s \ \& \sim N(0, t-s)\right).$

$\Rightarrow (M, N)$ is a 2D B.M.

Q3: Market

- $\to$ Stock $S(t) \to GBM(\alpha, \sigma)$
- $\to$ M.M. ctsly compounded interest rate r

Derivative security pays $V(T) = \frac{1}{T}\int_0^T S(t)\,dt$ at time T

(A) Compute the arbitrage free price at time $t < T$.

Sol: RNP formula: $\boxed{\tau = T - t}$

Know $V(t) = \tilde{E}\left(e^{-r\tau} V(T) \,\middle|\, \mathcal{F}_t\right)$

& Know under $\tilde{P}$, $S = GBM(r, \sigma)$

Compute: $\tilde{E}\left(\frac{e^{-r\tau}}{T}\int_0^T S(s)\,ds \,\middle|\, \mathcal{F}_t\right)$

$$= \frac{e^{-r\tau}}{T}\tilde{E}\left(\int_0^t S(s)\,ds + \int_t^T S(s)\,ds \,\middle|\, \mathcal{F}_t\right)$$

$$= \frac{e^{-r\tau}}{T}\left(\int_0^t S(s)\,ds + \int_t^T \tilde{E}\left(S(s)\,\middle|\,\mathcal{F}_t\right) ds\right) \cdots (*) .$$

$(s > t)$

Know $S(s) = S(0)\exp\left(\left(r - \frac{\sigma^2}{2}\right)s + \sigma\widetilde{W}(s)\right) \qquad (s > t)$

$$= S(t)\exp\left(\left(r - \frac{\sigma^2}{2}\right)(s-t) + \sigma\left(\widetilde{W}(s) - \widetilde{W}(t)\right)\right).$$

$$\Rightarrow \widetilde{E}\left(S(s) \mid \mathcal{F}_t\right) = \widetilde{E}\left(S(t)\exp\left(\left(r - \frac{\sigma^2}{2}\right)(s-t) + \sigma\left(\widetilde{W}(s) - \widetilde{W}(t)\right)\right) \Big| \mathcal{F}_t\right).$$

$$= S(t)\exp\left(\left(r - \frac{\sigma^2}{2}\right)(s-t)\right)\widetilde{E}\left(e^{\sigma\left(\widetilde{W}(s) - \widetilde{W}(t)\right)} \Big| \mathcal{F}_t\right).$$

$$= S(t)\exp\left(\left(r - \frac{\sigma^2}{2}\right)(s-t) + \frac{\sigma^2}{2}(s-t)\right)$$

$$= S(t)\, e^{r(s-t)}$$

$$\Rightarrow V(t) = \frac{e^{-r\tau}}{T}\left(\int_0^t S(s)\,ds + \int_t^T S(t)\, e^{r(s-t)}\,ds \right).$$

$$= \frac{e^{-r\tau}}{T}\left(\int_0^t S(s)\,ds + \frac{S(t)}{r}\left(e^{r(T-t)} - 1 \right) \right).$$

$$\boxed{V(t) = \frac{e^{-r\tau}}{T}\int_0^t S(s)\,ds + S(t)\left(\frac{1 - e^{-r\tau}}{rT} \right).}$$

Compute R. Portfolio. $V(t) = X(t) \begin{cases} \to \Delta(t) \text{ shares of } S \\ \to \text{Rest cash.} \end{cases}$

$$\Delta(t) = \frac{1 - e^{-r(T-t)}}{rT}.$$ Note $\Delta(T) = 0$

$\boxed{4.8:}$ $$B(t) = \int_0^t \operatorname{sign}(W(s))\, dW(s) \qquad \operatorname{sign}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

① Is B a BM. (Yes → Levy. B is a mg & $d[B,B] = 1\, dt \Rightarrow$ QED.)

② Is $W(t) = \int_0^t \sigma(s)\, dB(s)$ for some σ?

$$dB(s) = \operatorname{sign}(W(s))\, dW(s) \Rightarrow dW(s) = \operatorname{sign}(W(s))\, dB(s)$$

$$\boxed{\Rightarrow W(t) = \int_0^t \operatorname{sign}(W(s))\, dB(s).}$$

③ Compute $[B, W]$ =

$$\left.\begin{aligned} W(t) &= \int_0^t 1 \, dW \\ B(t) &= \int_0^t \text{sign}(W(s)) \, dW(s) \end{aligned}\right\} \Rightarrow \boxed{[B, W](t) = \int_0^t \text{Sign}(W(s)) \, ds}$$

④ Are B, W ind? $\boxed{\text{Uncorrelated?}}$

Compute: $E(BW)$: $d(BW) = B\,dW + W\,dB + \text{Sign}(W(t))\,dt$

$$\Rightarrow E\,B(t)\,W(t) = 0 + 0 + \int_0^t E\,\text{Sign}(W(s))\,ds = 0$$

$\Rightarrow B(t)$ & $W(t)$ are uncorrelated.

B & W are norml $\Longrightarrow$ ~~B & W are ind?~~

(But not Joint Norml) $\not\Longrightarrow$ B & W are ind!

Know B & W are NOT ind because

$$[B, W] = \int_0^t \operatorname{Sign}(W(s))\,ds \neq 0.$$

$\boxed{4.5:}$ $dX = \theta(\mu - X(t))\,dt + \sigma\,dW$

$\theta, \mu, \sigma \in \mathbb{R}$.

① Solve & find a formula for X.

Set $Y(t) = e^{\theta t} X(t)$

$$\Rightarrow dY = \theta e^{\theta t} X\,dt + e^{\theta t}\,dX$$

$$= \theta Y\,dt + e^{\theta t}\Big(\theta(\mu - X(t))\,dt + \sigma\,dW\Big)$$

$$= \cancel{\theta Y\,dt} + \theta e^{\theta t}\mu\,dt - \cancel{\theta Y\,dt} + \sigma e^{\theta t}\,dW.$$

$$= \theta e^{\theta t}\mu\,dt + \sigma e^{\theta t}\,dW$$

$$\Rightarrow Y(t) - Y(0) = \mu\left(e^{\theta t} - 1\right) + \sigma \int_0^t e^{\theta s}\, dW(s)$$

Let $x_0 = X(0)$.

$$\Rightarrow X(t) = e^{-\theta t} Y(t)$$

$$= e^{-\theta t}\left(x_0 + \mu\left(e^{\theta t} - 1\right) + \sigma \int_0^t e^{\theta s}\, dW(s)\right).$$

$$\boxed{X(t) = e^{-\theta t} x_0 + \mu\left(1 - e^{-\theta t}\right) + \sigma e^{-\theta t} \int_0^t e^{\theta s}\, dW(s)}$$

Ornstein Uhlenbeck Process.