$\boxed{\text{Rec 6}}$

Today! - ~~Ito~~ Multivariate Ito
- Change of Measure + Girsanov
- RNP.

Recall: Ito in N-D

$$f(t, X_1 \dots X_n) = f(t, \underline{X})$$

if $f \in C^{1,2}$ then we have Ito's formula:

$\underline{X} = (X_1, \dots X_n)$.

X_t Ito process $\rightarrow$ N-Dimensional

$$df(t, X(t)) = \partial_t f(t, X(t))\, dt + \sum_{i=1}^{n} \partial_i f(t, X_i) . dX_i(t)$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \partial_i \partial_j f(t, X(t))\, d[X_i, X_j](t).$$

Ex 1

$$\begin{cases} dX = \mu_1 X\,dt + \sigma_1 X\,dW_1(t) \\ dY = \mu_2 Y\,dt + \sigma_2 Y\,dW_2(t) \\ \quad d[W_1, W_2] = \rho\,dt \quad , -1 \le \rho \le 1 \end{cases}$$

$Z_t = \frac{X_t}{Y_t}$ find dZ_t

$f(t,x,y) = \frac{x}{y}$ $\quad f_t = 0 \quad f_x = \frac{1}{y}$, $f_{xx} = 0$.

$f_y = \frac{-x}{y^2}$, $f_{yy} = \frac{2x}{y^3}$.

$f_{xy} = f_{yx} = \frac{-1}{y^2}$.

$$df(t,x,y) = \frac{1}{y}dx - \frac{x}{y^2}dy + \frac{1}{2}\left[\frac{2x}{y^3}d[y,y]\right] + \frac{1}{2}(2)\left(\frac{-1}{y^2}d[x,y]\right) + 0.$$

$$= \frac{1}{y}dx - \frac{x}{y^2}dy + \frac{x}{y^3}(\sigma^2 y^2)dt \quad \frac{-1}{y^2}d[x,y] + 0.$$

$$[X, Y] = [I_1 + B_1, I_2 + B_2].$$

$$\overset{\text{Bi-linearity}}{=} [I_1 + B_1, I_2] + \underbrace{[I_1 + B_1, B_2]}_{0}.$$

$$= [I_1, I_2] + \underbrace{[B_1, I_2]}_{0}.$$

$$= [I_1, I_2]$$

J_1

Prop 9.10 in Your notes. !

$$\boxed{\begin{array}{l} I_j(t) = \int_0^t \sigma_j(s)\, d X_j(s), \quad j = 1, 2 \\ \text{then } [I_1, I_2] = \int_0^t \sigma_1 \sigma_2 \, d[X_i, X_j](s) \end{array}}$$

$$\therefore [X, Y] = \int_0^t \sigma_1 X \sigma_2 Y \, d[W_1, W_2]s = \int_0^t \sigma_1 \sigma_2 X Y \rho \, dt.$$

$$d[X, Y] = \rho \sigma_1 \sigma_2 X Y \, dt.$$

$$\therefore\ d(f) = \frac{1}{Y}\,dX - \frac{X}{Y^2}\,dY - \frac{1}{Y^2}(\rho\sigma_1\sigma_2 XY\,dt) + \frac{1}{2}\sigma_2^2 Y^2\left(\frac{2X}{Y^3}\right)dt$$

$$dZ_t = \frac{\mu_1 X_t d_t + \sigma_1 X dW_1(t)}{Y_t} - \frac{[X(\mu_2 Y_t d_t + \sigma_2 Y dW_2(t))]}{Y_t^2}$$

$$- \rho\sigma_1\sigma_2 \frac{X}{Y}\,dt + \sigma_2^2 \frac{X}{Y}\,dt.$$

$$\boxed{= \underbrace{\frac{X_t}{Y_t}}_{Z}\left[[\mu_1 - \mu_2 - \rho\sigma_1\sigma_2 + \sigma_2^2]\,dt + \sigma_1\,dW_1(t) - \sigma_2\,dW_2(t)\right]}$$

$$\frac{dZ_t}{Z} = (\mu_1 - \mu_2 - \rho\sigma_1\sigma_2 + \sigma_2^2)dt + (\sigma_1\,dW_1(t) - \sigma_2\,dW_2(t)).$$

$$\left(\frac{\partial Z_t}{Z}\right)^2 \approx o(d_t^2) + \underbrace{o(\partial_t \partial W_1)}_{\partial_t^{3/2}} + \underbrace{o(\partial_t \partial W_2)}_{d_t^{3/2}} + (\sigma_1\,dW_1(t) - \sigma_2\,dW_2(t))^2$$

$$W_t^2 \approx d_t.$$

so in "small time" $(\sigma_1\,dW_1(t) - \sigma_2\,dW_2(t))^2$ dominates

$$(\sigma_1 \, dW_1(t) - \sigma_2 \, dW_2(t))^2 \approx \sigma_1^2 \, dt + \sigma_2^2 \, dt - 2\sigma_1\sigma_2 \, d[W_1, W_2](t).$$

$$\approx \underbrace{(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)}_{\boxed{\sigma_*} \text{ from the Magrabe formula}} dt$$

we discussed last time.

This is called the instantaneous volatility wrt X, Y.

<u>Girsanov Thm.</u> $\underline{\theta} = (\theta_1, \dots \theta_d)$ adapted process.

$\underline{W} = (W_1, \dots W_d)$ d-Dimensional B-M.

$$Z(t) = \exp\left(-\int_0^t \theta(s)\,dW_s - \frac{1}{2}\int_0^t |\theta(s)|^2 \, ds\right).$$

Then $\tilde{W} = W_t + \int_0^t \theta(s)\,ds = \left(W_1(t) + \int_0^t \theta_1(s)\,ds, \dots, W_d(t) + \int_0^t \theta_d(s)\,ds\right)$

is a BM under $\tilde{P}$, where $d\tilde{P} = Z(t)\,dP$.

Critical Thm for RNP + FTAP.

Ex. $X \sim N(0,1)$. (under P)

Define $d\tilde{P} = \underbrace{e^{\alpha X + \beta}}_{Z} dP$.

Find α, β, a s.t. $X + a \sim N(0,1)$ under $\tilde{P}$.

Certainly need $E[Z] = 1$.

By MGF formula $E[Z] = e^{\beta + \frac{\alpha^2}{2}} \overset{\text{set}}{=} 1$.

need $\boxed{\beta = -\frac{\alpha^2}{2}}$.

$$Z = e^{\alpha X - \frac{\alpha^2}{2}}.$$

Recall $W_1 \sim N(0,1)$, $Z = \exp\left(-\int_0^1 -\alpha \, dW - \int_0^1 \frac{(-\alpha)^2}{2} ds\right)$

By Girsanov with $t=1$, $d=1$, $\theta(t) = -\alpha$.

we have that $\tilde{X} \overset{d}{=} \tilde{W}_1 = W_1 + \int_0^1 -\alpha \, ds = W_1 - \alpha \overset{d}{=} X - \alpha$

is $N(0,1)$ under $\tilde{P}$. (i.e. set $a = -\alpha$).

Let's verify. $Y = X - \alpha$

$$\tilde{P}(Y \le y) = \int_{\{Y \le y\}} Z\,dP = \int_{-\infty}^{\infty} \mathbb{I}_{\{Y \le y\}} e^{\alpha X - \frac{\alpha^2}{2}} \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{X^2}{2}}\right) dx$$

$$Y \le y \iff X \le y + \alpha$$

$$= \int_{-\infty}^{y+\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(X^2 - 2\alpha X + \alpha^2)} dx$$

$$= \int_{-\infty}^{y+\alpha} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(X-\alpha)^2} dx$$

$z = X - \alpha$. $\quad X \to -\infty \Rightarrow z \to -\infty$

$dz = dx$. $\quad X \to y + \alpha \Rightarrow z \to y$.

$$= \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = N(y).$$

Ex using $d[I_1, I_2] = \int_0^t \sigma_1(s, X_1(s))\, \sigma_2(s, X_2(s))\, d[W_1, W_2](s)$.

we can prove Ito Isometry.

set $\sigma_1 = \sigma_2$

$$Y^2 = \left(\int_0^t \sigma(s, X(s))\, dW_s\right)^2 = \left(\int_0^t \sigma_1(s, X(s))\, dW_s\right)\left(\int_0^t \sigma(s, X(s))\, dW_s\right)$$

$$d(MN) = M\,dN + N\,dM + d[M,N]$$

so $d(Y^2) = \underline{2Y\,dY} + d[Y,Y]$.

$$E[Y^2] = E[Y,Y] = E\left[\int_0^t \sigma_1^2\, dt\right]$$