Last time: CM-Girsanov:

$$d\tilde{W} = b\,dt + dW \qquad (W \to d \text{ dim B.M})$$

Define $Z$ by $\quad dZ = -Z\, b \cdot dW$

$$\left( Z(t) = \exp\left( -\int_0^t b \cdot dW - \frac{1}{2}\int_0^t |b|^2\, ds \right)\right).$$

$\tilde{P} \to$ new measure: $T \to$ time Horizon (maturity time).

$$d\tilde{P} = Z(T)\, dP$$

Thm: $\tilde{W}$ is a std d-dim BM under $\tilde{P}$

Very end of last time: $\qquad d$

$$d[\tilde{W}_i, Z] = ?$$

Know $d\tilde{W}_i(t) = b_i(t)\,dt + dW_i(t)$

& $dZ = -Z\, b \cdot dW = -Z \sum_{j=1}^{d} b_j\, dW_j$

$$\Rightarrow \cancel{d\tilde{W}_i} \boxed{d[\tilde{W}_i, Z] = -Z\, b_i\, dt}$$

$$\left(\because d[W_i, W_j] = 1_{\{i=j\}}\, dt\right).$$

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If $dX = \sigma\, dM + (\;)\,dt$
& $dY = \tau\, dN + (\;)\,dt$ $\Big\} \Rightarrow d[X,Y] = \sigma\tau\, d[M,N]$.

Risk Neutral Pricing:

Stock Price $S(t) \longrightarrow GBM(\alpha, \sigma)$

$$dS = \alpha S\, dt + \sigma S\, dW$$

Investor $\begin{cases} \to \text{Stock } S \\ \to \text{M.M. ctsly compounded interest rate } R(t) \end{cases}$ ← rate

($R(t) \longrightarrow$ adapted process).

Discount Process: $D(t) = \exp\left(-\int_0^t R(s)\, ds\right)$.

$$\boxed{dD = -R\, D\, dt}$$

Useful (mathematically) $\longrightarrow$ Discounted stock Price : $D(t)\,S(t)$.

<u>Def:</u> (Risk Neutral Measure). The risk neutral measure is an equivalent measure $\tilde{P}$ under which $DS$ is a mg.

$(DS = D(t)S(t) = \text{discounted Stock price})$.

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RNM (one asset + MM):

$$d(DS) = D\,dS + S\,dD + \overset{0}{\overbrace{d[D,S]}}$$

$$= -RSD\,dt + D(\alpha S\,dt + \sigma S\,dW)$$

$$= \big((\alpha - R)SD\big)dt + \sigma SD\,dW$$

$$= \sigma SD\left(\left(\frac{\alpha - R}{\sigma}\right)dt + dW\right).$$

Let $\theta(t) = \dfrac{\alpha - R(t)}{\sigma}$ (called the "Market Price of risk").

$$\Rightarrow d(DS) = \sigma S D \underbrace{(\theta\, dt + dW)}_{d\tilde{W}}.$$

Let $\tilde{W}(t) = \int_0^t \theta(s)\, ds + W(t)$ (i.e $d\tilde{W} = \theta\, dt + dW$).

<u>Girsanov</u>: $Z(t) = \exp\left(-\int_0^t \theta(s)\, dW - \frac{1}{2}\int_0^t \theta(s)^2\, ds\right)$.

Fix $T > 0$. $\boxed{\text{Set } d\tilde{P} = Z(T)\, dP} \leftarrow$ Risk Neutral Measure.

Know $\tilde{W}$ is a B.M. under $\tilde{P}$ ($\Rightarrow \tilde{W}$ mg under $\tilde{P}$).

$\Rightarrow d(DS) = \sigma S D \, d\tilde{W}$, which $\underline{\underline{\text{IS}}}$ a mg under $\tilde{P}$!

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<u>Theorem</u>: (Risk Neutral Pricing Formula).

$V(T) \longrightarrow \mathcal{F}_T$ meas  Payoff of a security

Let $\tilde{P}$ be the RNM.

The arbitrage free price of this security at any time $t \leq T$ is given by

$$V(t) = \text{AFP at time } t = \tilde{E}\left( \exp\left(-\int_t^T R(s)\,ds\right) V(T) \,\Big|\, \mathcal{F}_t \right)$$

<u>Remark</u>: ① Existence of RNM $\iff$ No arbitrage

& ② Uniqueness of RNM $\iff$ No arbitrage & every derivative security can be hedged.

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Main Reason for RNP formula:

$$dS = \alpha S\,dt + \sigma S\,dW = \alpha S\,dt + \sigma S\left(d\widetilde{W} - \theta\,dt\right)$$

$$= \alpha S\,dt + \sigma S\,d\widetilde{W} - S(\alpha - R)\,dt$$

$$= RS\,dt + \sigma S\,d\widetilde{W}.$$

<u>Lemma</u>: Let $\Delta$ be any adapted process.

$X(t)$ = wealth of an investor holds $\Delta(t)$ shares of stock & rest in cash.

If there is no external cash flow (i.e. pf is self financing).

then $DX$ is a mg under $\tilde{P}$

<u>Pf:</u> Compute $d(DX) = D\,dX + X\,dD + d[X, D]$

$$d[X,D] = 0$$

$$= -RDX\,dt + D\big(\Delta\,dS + R(X - \Delta S)\,dt\big)$$

$$\big(\because dX = \Delta\,dS + R(X - \Delta S)\,dt\big).$$

$$= -\Delta SRD\,dt + D\Delta\big(RS\,dt + \sigma S\,d\tilde{W}\big)$$

$$= D\Delta\sigma S\,d\tilde{W} \Rightarrow DX \text{ is a mg under } \tilde{P}$$

QED.

Proof of RNP formula: Security $\to$ payoff $V(T)$.

Let $X(t)$ = value of replicating portfolio at time $t$.

AFP $\Rightarrow$ AFP of sec at time $t = V(t) = X(t)$.

Knows $DX$ is a mg under $\tilde{P}$!

$$\Rightarrow V(t) = X(t) = \frac{1}{D(t)}\left(D(t)X(t)\right)$$

$$= \frac{1}{D(t)}\tilde{E}\left(D(T)X(T) \mid \mathcal{F}_t\right) \quad (\because DX \text{ is a } \tilde{P} \text{ mg}).$$

$$= \tilde{E}\left(\frac{D(T)}{D(t)}V(T) \mid \mathcal{F}_t\right) = \tilde{E}\left(\exp\left(-\int_t^T R(s)\,ds\right)\cdot V(T) \mid \mathcal{F}_t\right)$$

QED

Rmk: Pf assumes existence of a R. Portfolio.

Mg representation theorem: Any mg that is adapted to the filtration generated by BM is also an Itô int wrt the BM.

Let $Y(t) = \tilde{E}\left( \exp\left( -\int_t^T R(s)\,ds \right) V(T) \,\Big|\, \mathcal{F}_t \right)$

= mg (wrt $\tilde{P}$) & adapted to filtration of $\tilde{W}$.
↑
BM.

Use mg rep thm, write as ito int & can get existence of a R-Pf.

Rmk: Say $V(T) = f(S(T))$

Markov Property: $\tilde{E}\left( \underline{\hspace{4em}} \mid \mathcal{F}_t \right) = c(t, S(t))$

for some function $c$.

R. Portfolio

Set $c(t, S(t)) = X(t)$ & Itô

$$\boxed{\Rightarrow \text{Delta Hedging Rule : } \Delta(t) = \partial_x c(t, S(t))}$$

Derive Black Scholes formula (Using RNP formula).

$S \longrightarrow GBM(\alpha, \sigma)$.

$R(t) = r$ (independent of time).

European call maturity $T$, strike $K$.

Payoff $V(T) = (S(T) - K)^+$

Let $c(t, S(t))$ = price of this call at time $t$.

$$\stackrel{\text{RNPF}}{=} \widetilde{E}\left( e^{-r(T-t)} (S(T)-K)^+ \,\middle|\, \mathcal{F}_t \right).$$

~~① $S(t) =$~~ (Bad way: $\tilde{E}(X \mid \mathcal{F}_t) = \frac{1}{Z(t)} E(Z(T)X \mid \mathcal{F}_t)$

will work, but too long).

Almost ~~Good~~ way: $S(t) = S(0) \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right)$

Under $\tilde{P}$, $W$ is not explicitly known. (not nice).

$$\boxed{S(t) = S(0) \exp\left(\left(r - \frac{\sigma^2}{2}\right)t + \sigma \tilde{W}(t)\right)}$$

($S$ is GBM with mean return rate $r$ under $\tilde{P}$).

$$\Rightarrow c(t, S(t)) = \widetilde{E}\left( e^{-r(T-t)} (S(T)-K)^+ \,\middle|\, \mathcal{F}_t \right).$$

$$\boxed{\tau = T - t}$$

$$= \widetilde{E}\left( \left[ S(0) \exp\left( \left(r - \frac{\sigma^2}{2}\right) T + \sigma \widetilde{W}(T) \right) - K \right]^+ \,\middle|\, \mathcal{F}_t \right) e^{-r\tau}$$

$$= e^{-r\tau} \widetilde{E}\left( \left[ S(t) \exp\left( \left(r - \frac{\sigma^2}{2}\right) \tau + \sqrt{\tau}\,\sigma \left( \frac{\widetilde{W}(T) - \widetilde{W}(t)}{\sqrt{\tau}} \right) \right) - K \right]^+ \,\middle|\, \mathcal{F}_t \right)$$

Ind lemma

$$= e^{-r\tau} \int_{\mathbb{R}} \left( S(t) \exp\left( \left(r - \frac{\sigma^2}{2}\right) \tau + \sigma \sqrt{\tau}\, y \right) - K \right)^+ \frac{e^{-y^2/2}}{\sqrt{2\pi}} \, dy$$

& simplify this integral to get Black Scholes.