Last time: $df(t, X(t)) = \partial_t f \, dt + \partial_i f \, dX_i(t)$

$X_1, X_2, \dots X_d$

$+ \frac{1}{2} \sum_{i,j=1}^{d} \partial_i \partial_j f \, d[X_i, X_j](t)$

Multi D B.M: $W = (W_1, W_2, \dots W_d)$

① Each W_i is a ~~1D~~ standard 1D BM.

& ② For $i \neq j$, W_i is ind of W_j.

Filtration: $\mathcal{F}_t^W = \sigma\left(\bigcup_{i=1}^{d} \bigcup_{s \le t} \sigma(W_i(s)) \right)$

Note: W_i & W_j are ind for $i \neq j$, $[W_i, W_j] = 0$

(altente notation "$dW_i \, dW_j$" $= 0$).

$\to$ for $i = j$ $d[W_i, W_i](t) = dt$.

$\Rightarrow d[W_i, W_j](t) = \mathbb{1}_{\{i=j\}} \, dt$

<u>Thm (Levy)</u>: If M is a <u>cts</u> <u>mg</u>. such that

$$d[M_i, M_j] = \mathbb{1}_{\{i=j\}} \, dt$$

then M is a standard d-dim BM.

Eg from last time: $W \to$ 2D BM.

$$f(x) = f(x_1, x_2) = \ln |x|^2 = \ln(x_1^2 + x_2^2).$$

$$\partial_1 f = \frac{1}{|x|^2} \cdot 2x_1, \quad \partial_2 f = \frac{2x_2}{|x|^2}$$

& You compute $\Delta f = \partial_1^2 f + \partial_2^2 f = 0$

Set $Y(t) = f(W(t)) = f(W_1(t), W_2(t)) = \ln\left(W_1(t)^2 + W_2(t)^2\right)$.

Compute
$$dY = \frac{2W_1(t)}{|W(t)|^2} dW_1(t) + \frac{2W_2(t)}{|W(t)|^2} dW_2(t)$$
$$+ \frac{1}{2} \cdot 0 \cdot dt.$$

Guess Y is a mg.

But Y can not be a mg.

Know, if Y is a mg, then $EY(t) = EY(0)$ ← constant in time.

Compute $EY(t) = E \ln(|W(t)|^2)$

$$= \int_{\mathbb{R}^2} \ln(|x|^2) \frac{1}{2\pi t} e^{-|x|^2/2t} \, dx_1 \, dx_2 .$$

$y = \frac{x}{\sqrt{t}}$

$dx_1 \, dx_2 = dy \cdot t$

$$= \int_{\mathbb{R}^2} \ln(t|y|^2) \, e^{-|y|^2/2} \, \frac{dy_1 \, dy_2}{2\pi} .$$

$$= \ln t \underbrace{\int_{\mathbb{R}^2} (\ ) \, dy_1 \, dy_2}_{=1} + \int_{\mathbb{R}^2} \ln|y^2| \, e^{-|y|^2/2} \, dy_1 \, dy_2 .$$

$$= \ln t + \int_{\mathbb{R}^2} \underbrace{(\quad)}_{\text{doesn't dep on } t} dy_1\, dy_2.$$

$=$ not constant in time.

Reason: $\int_0^t \sigma(s)\, dW$ is only guaranteed to be a mg

if $E \int_0^t \sigma(s)^2 ds < \infty$.

Risk Neutral Measures:

Security $\to$ payoff $V(T)$ ($\mathcal{F}_T$ measurable).

RNPF says ~~price~~ AFP of this security is.

$$\tilde{E}\left(V(T)\, e^{-r(T-t)} \,\middle|\, \mathcal{F}_t \right).$$

$\tilde{E}$ = cond exp wrt the RNM.

(IOU, today).

Def: Two ~~pr~~ measures P & $\tilde{P}$ are said to be equivalent if ~~whenever~~ $P(A) = 0 \Leftrightarrow \tilde{P}(A) = 0$

Eg: Let Z be a r.v, such that ① $EZ = 1$ & ② $Z > 0$

Define $\tilde{P}(A) = \int_A Z \, dP$

Then ~~P~~ ~~&~~ ~~P~~ $\tilde{P}$ is a pr measure

& P & $\tilde{P}$ are equiv.

Thm: If P & $\tilde{P}$ are equiv, then ~~∃~~ there exists a RV Z such that $\tilde{P}(A) = \int_A Z \, dP$ for all measurable sets A.

(Radon Nicodym thm).

Notation: If $\tilde{P}$ is defined by $\tilde{P}(A) = \int_A Z \, dP$ for all A, then

① We say $d\tilde{P} = Z \, dP$

& ② Z is called the Radon Nycodym derivative of $\tilde{P}$ wrt P.

$\left(Z = \frac{d\tilde{P}}{dP} \right)$.

Remark: If X is a R.V., then denote $\tilde{E}X$ = Exp of X wrt $\tilde{P}$

If $d\tilde{P} = Z \, dP$, then $\tilde{E}X = \cancel{Z E X} \quad E(XZ)$.

$$= \int X \, d\tilde{P} = \int XZ \, dP$$

<u>Thm</u> (Cameron Martin Girsanov). Let $b = (b_1, b_2, \dots b_d)$ some d-dimensional adapted <u>process</u>.

$W \longrightarrow$ d - dim B.M.

Let $\tilde{W}(t) = W(t) + \int_0^t b(s)\, ds$. $\left(d\tilde{W} = b\, dt + dW \right)$.

Let $Z(t) = \exp\left(- \int_0^t b(s) \cdot dW(s) - \frac{1}{2} \int_0^t |b(s)|^2 ds \right)$.

Fix $T > 0$. & define a new measure $\tilde{P} = \tilde{P}_T$ by

$d\tilde{P} = Z(T)\, dP$.

If Z is a mg (under P), then $\tilde{W}$ is a BM wrt $\tilde{P}$ (up to time T).

<u>Rmk:</u> $b \cdot dW = \sum_{i=1}^{d} b_i \, dW_i$

<u>Rmk:</u> Compute dZ. Let $M(t) = \int_0^t b(s) \cdot dW(s) = \sum_1^d \int_0^t b_i(s) \, dW_i(s)$

Note $[M,M](t) = \sum_1^d \int_0^t b_i^2(s) \, ds. = \int_0^t |b(s)|^2 \, ds.$

Let $f(t,x) = e^{-x - \frac{1}{2}\int_0^t |b(s)|^2 ds} \Rightarrow Z(t) = f(t, M(t)).$

$$\Rightarrow dZ = \partial_t f \, dt + \partial_x f \, dM + \frac{1}{2} \partial_x^2 f \, d[M,M].$$

$$= -\frac{1}{2} \exp(\quad) |b(t)|^2 dt - Z \, dM + \frac{1}{2} Z \, d[M,M].$$

$$= -\frac{1}{2} Z |b|^2 dt - Z \, dM + \frac{1}{2} Z |b(t)|^2 dt =$$

$= -Z\, \sigma \cdot dW$

$\Rightarrow$ Z is a mg (PROVIDED $E\int_0^t Z^2 |\sigma|^2 ds < \infty$).

Dont know.

Need Z to be a mg because:

we need $\tilde{P}$ to be a pr measure.

Note $\tilde{P}(\Omega) = E\, Z(T)$

If Z is a mg, then $E\, Z(T) = E\, Z(0) = 1$.

Strategy of Proof: ① Check $\tilde{W}$ is a mg under $\tilde{P}$ } & use Lévy'
② Check $d[\tilde{W}_i, \tilde{W}_j] = \mathbb{1}_{i=j}\, dt$ }.

$\underline{\text{Lemma}}$: Let X be a $\mathcal{F}_t$ meas RV. $0 \leq s \leq t$.

$$\text{then } \tilde{E}(X \mid \mathcal{F}_s) = \frac{1}{Z(s)} E(Z(t) X \mid \mathcal{F}_s).$$

cond exp wrt $\tilde{P}$

Proof: Let $A \in \mathcal{F}_s$.

$$\int_A \tilde{E}(X \mid \mathcal{F}_s)\, d\tilde{P} = \int_A \tilde{E}(X \mid \mathcal{F}_s) \cdot Z(T)\, dP \qquad (\because d\tilde{P} = Z\, dP).$$

$$= \int_A E\left(Z(T)\, \tilde{E}(X \mid \mathcal{G}_s) \;\middle|\; \mathcal{G}_s \right) dP$$

$$= \int_A \tilde{E}(X \mid \mathcal{G}_s)\, E\left(Z(T) \mid \mathcal{G}_s \right) dP$$

$$= \int_A \tilde{E}(X \mid \mathcal{G}_s)\, Z(s)\, dP. \quad \ldots \quad (1)$$

Also, $\displaystyle \int_A \tilde{E}(X \mid \mathcal{G}_s)\, d\tilde{P} = \int_A X\, d\tilde{P} = \int_A X\, Z(T)\, dP$

$$= \int_A E\left(X\, Z(T) \mid \mathcal{G}_t \right) dP = \int_A X\, Z(t)\, dP$$

$$= \int_A E\left(X\, Z(t) \mid \mathcal{G}_s \right) dP \quad \ldots \quad (2)$$

Lemma 2: An adapted process M is a mg under $\widetilde{P}$

$\iff$ MZ is a mg under P.

Pf: (1) Say MZ is a mg under P.

Compute
$$\widetilde{E}\left(M(t) \mid \mathcal{F}_s\right) \overset{\text{Lemma 1}}{=} \frac{1}{Z(s)} E\left(M(t) Z(t) \mid \mathcal{F}_s\right)$$

$$\overset{MZ \text{ a mg}}{=} \frac{1}{Z(s)} M(s) Z(s) = M(s)$$

QED.

(2) Conversely, say M is a mg under $\widetilde{P}$.

Compute
$$E\left(M(t) Z(t) \mid \mathcal{F}_s\right) = Z(s)\, \widetilde{E}\left(M(t) \mid \mathcal{F}_s\right) = Z(s) M(s)$$

QED.

$\Rightarrow$ for <u>every</u> $A \in \mathcal{F}_s$,

$$\int_A \tilde{E}(X \mid \mathcal{F}_s)\, Z(s)\, dP = \int_A E(X Z(t) \mid \mathcal{F}_s)\, dP.$$

$$\Rightarrow \tilde{E}(X \mid \mathcal{F}_s) \cdot Z(s) = E(X Z(t) \mid \mathcal{F}_s).$$

$$\boxed{\Rightarrow \tilde{E}(X \mid \mathcal{F}_s) = \frac{1}{Z(s)} E(X Z(t) \mid \mathcal{F}_s).}$$

Proof of Girsanov:

Claim ①: $d[\tilde{W}_i, \tilde{W}_j] = \mathbb{1}_{\{i=j\}} \, dt$

Pf: $d[\tilde{W}_i, \tilde{W}_j] = d[W_i, W_j] = \mathbb{1}_{\{i=j\}} \, dt$.

Claim ②: $\tilde{W}_i$ is a mg under $\tilde{P}$.

Pf: By Lemma 2, only NTS $\tilde{W}_i Z$ is a mg under P.

Check: Compute $d(\tilde{W}_i Z) = \tilde{W}_i \, dZ + Z \, d\tilde{W}_i + d[\tilde{W}_i, Z]$.

$$= - Z \tilde{W}_i \, b \cdot dW + Z(b_i \, dt + dW) - Z b_i \, dt$$

dt terms cancel! $\Rightarrow$ $\tilde{W}_i Z$ is a mg under P.

By Levy $\Rightarrow$ $\tilde{W}$ is a BM under $\tilde{P}$.

QED.

Proof of Girsanov:

Claim ①: $d[\tilde{W}_i, \tilde{W}_j] = \mathbb{1}_{\{i=j\}} \, dt$

Pf: $d[\tilde{W}_i, \tilde{W}_j] = d[W_i, W_j] = \mathbb{1}_{\{i=j\}} \, dt$.

Claim ②: $\tilde{W}_i$ is a mg under $\tilde{P}$.

Pf: By Lemma 2, only NTS $\tilde{W}_i Z$ is a mg under P.

Check: Compute $d(\tilde{W}_i Z) = \tilde{W}_i \, dZ + Z \, d\tilde{W}_i + d[\tilde{W}_i, Z]$.

$$= -Z\tilde{W}_i \, b \cdot dW + Z(b_i \, dt + dW) - Z b_i \, dt$$

dt terms cancel! $\Rightarrow$ $\tilde{W}_i Z$ is a mg under P.

By Levy $\Rightarrow$ $\widetilde{W}$ is a BM under $\widetilde{P}$.

QED.