Last time: Joint QV.

$X, Y \to 2$ processes. $P = \{0 = t_0 < t_1 < \dots t_n = T\}$.

$$[X,Y](T) \stackrel{\text{def}}{=} \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} (\Delta_i X)(\Delta_i Y)$$

$$\Delta_i X = X_{t_{i+1}} - X_{t_i}, \quad \Delta_i Y = Y_{t_{i+1}} - Y_{t_i}$$

Last time: $\underbrace{[X, Y]}_{\text{J. QV}} = \frac{1}{4}\Big(\underbrace{[X+Y, X+Y]}_{\text{regular QV}} - \underbrace{[X-Y, X-Y]}_{\text{regular QV}}\Big)$

<u>Product Rule:</u> $d(XY) = X\,dY + Y\,dX + \underbrace{d[X,Y]}_{\text{extra}}$

Multi D Ito:

$$X = (X_1, X_2, \cdots X_n) \qquad X_1, X_2, \cdots X_n \text{ are all S. processes.}$$

$$f = f(t, x) = f(t, \underbrace{x_1, \cdots x_n}_{x})$$

$f \to C^1$ in t & C^2 in x. ($\partial_t f$, $\partial_i f = \frac{\partial f}{\partial x_i}$ & $\partial_i \partial_j f$ all exist & are cts).

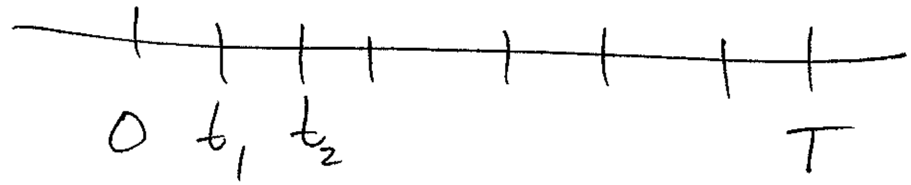
Then

$$d\big(f(t, X(t))\big) = \partial_t f \, dt + \sum_{1}^{n} \partial_i f \, dX_i(t) + \frac{1}{2} \sum_{i,j=1}^{n} \partial_i \partial_j f \, d[X_i, X_j](t)$$

(Note: $\partial_t f \, dt$ means $\partial_t f(t, X(t)) \, dt$ etc).

Intuition: fix T. Let $P = \{0 = t_0 < t_1, \dots t_m = T\}$

Simple case: Assume $n = 2$.



two processes X & Y.

& f doesn't depend on t.

$$f(X(T), Y(T)) - f(X(0), Y(0)) = \sum_{i=0}^{m-1} f(\xi_{i+1}) - f(\xi_i)$$

where $\xi_i = (X(t_i), Y(t_i))$.

Taylor Expand: $f(\xi_{i+1}) - f(\xi_i) = (\Delta_i X)\,\partial_x f(\xi_i) + \Delta_i Y\,\partial_y f$

$$+ \frac{1}{2}\Big((\Delta_i X)^2\,\partial_x^2 f(\xi_i) + (\Delta_i Y)^2\,\partial_y^2 f$$
$$+ 2(\Delta_i X)(\Delta_i Y)\,\partial_x\partial_y f \Big).$$

$$+ \text{Higher order terms.}$$

$$\Rightarrow f(X(T),Y(T)) - f(X(0),Y(0)) = \sum (\Delta_i X)\,\partial_x f(\xi_i) + (\Delta_i Y)\,\partial_y f(\xi_i).$$

$$+ \frac{1}{2}\sum (\Delta_i X)^2\,\partial_x^2 f + (\Delta_i Y)^2\,\partial_y^2 f + 2(\Delta_i X)(\Delta_i Y)\,\partial_x\partial_y f$$

$$+ \text{Higher order terms.}$$

take lim $\|P\| \to 0$:

Get $f(X(T),Y(T)) - f(X(0),Y(0)) = \int_0^T \partial_x f \; dX + \int_0^T \partial_y f \; dY$.

$$+ \frac{1}{2}\left(\int_0^T \partial_x^2 f \; d[X,X] + \int_0^T \partial_y^2 f \; d[Y,Y] + 2\int_0^T \partial_x \partial_y f \; d[X,Y] \right)$$

2D Itô formula in integral form.

QED.

QV: $M \to$ cts mg $\Rightarrow M^2 - [M,M]$ is a cts mg.

Conversely: If A is a cts increasing adapted process & $M^2 - A$ is a mg then $A = [M,M]$.

Joint QV: Suppose M, N are 2 cts mg's.

Assume $E M(t)^2 < \infty$ & $E N(t)^2 < \infty$.

Then (1) $MN - [M,N]$ is a mg.

(2) If A is any cts adapted process of finite first variation such that $MN - A$ is a mg, then $A = [M,N]$.

Proof: $4(MN - [M,N]) = (M+N)^2 - (M-N)^2 - ([M+N, M+N] - [M-N, M-N]).$

$$= (M+N)^2 - [M+N, M+N]$$
$$- \left((M-N)^2 - [M-N, M-N]\right).$$

$\Rightarrow MN - [M,N]$ is a mg.

Pf of (2). Known $MN - [M,N]$ is a mg.

& Given $MN - A$ is a mg

$\Rightarrow [M,N] - A$ is a mg.

But also, $[M,N] - A$ has finite first variation $\Big\} \Rightarrow [M,N] - A = 0$

QED.

(Note $[M,N]$ has finite first variation because:

$$[M,N] = \frac{1}{4}\left([M+N, M+N] - [M-N, M-N] \right)$$

= difference of two fns of finite first variation.

$\Rightarrow$ $[M,N]$ has finite first variation.

Prop: X, Y, Z are 3 Itô processes, $\alpha \in \mathbb{R}$

then $[X, Y+\alpha Z] = [X,Y] + \alpha [X, Z]$.

Pf: Itô decompose X, Y, Z.

Let L, M, & N be the mg parts of X, Y & Z resp'ly.

Know $[X, Y+\alpha Z] = [L, M+\alpha N]$.

Clearly $L(M+\alpha N) - ([L,M] + \alpha[L,N])$

$$= \underbrace{LM - [L,M]}_{\text{mg}} + \alpha(\underbrace{LN - [L,N]}_{\text{mg}}).$$

~~⇒~~ Also $[L,M] + \alpha[L,N]$ has finite 1st var.

$$\Rightarrow \quad [L, M+\alpha N] = [L,M] + \alpha[L,N].$$

$$\begin{array}{ccc} [L, M+\alpha N] & = & [L,M] + \alpha[L,N] \\ \| & & \| \\ [X, Y+\alpha Z] & & [X,Y] + \alpha[X,Z]. \end{array}$$

QED.

Prop: Let X_1, X_2 be two Itô processes.

σ_1, σ_2 be two adapted processes.

Let $I_1(t) = \int_0^t \sigma_1(s)\, dX_1(s)$ & $I_2(t) = \int_0^t \sigma_2(s)\, dX_2(s)$.

Then $[I_1, I_2](t) = \int_0^t \sigma_1(s)\, \sigma_2(s)\, d[X_1, X_2](s)$.

(Note: ~~thus~~ $[I_1, I_{\cancel{1}_2}](t) = \int_0^t \sigma_1(s)\, \sigma_{\cancel{1}_2}(s)\, d[X_1, X_{\cancel{1}_2}](s)$).

Proof: Partition P

$$I_1(T) = \lim_{\|P\| \to 0} \sum_{i=0}^{n-1} \sigma_1(t_i)\, \Delta_i X$$

Expect $\Delta_i I_1 \approx \sigma_1(t_i)\ \Delta_i X_1$

lll^{ly} $\Delta_i I_2 \approx \sigma_2(t_i)\ \Delta_i X_2$

$$\Rightarrow [I_1, I_2](T) = \lim_{\|P\|\to 0} \sum (\Delta_i I_1)(\Delta_i I_2)$$

$$= \lim_{\|P\|\to 0} \sum \sigma_1(t_i)\,(\Delta_i X_1)\,\sigma_2(t_i)(\Delta_i X_2)$$

$$= \int \sigma_1(s)\,\sigma_2(s)\; d\,[X_1, X_2](s)$$

QED.

Prop: Let M, N be 2 cts mg's. $EM(t)^2 < \infty$ & $EN(t)^2 < \infty$.

If M & N are independent, then $[M,N] = 0$

Remark: $[M,N] = 0 \not\Longrightarrow M, N$ are ind. (IOU Eg).

"Wrong Proof": If X, Y are ind $\Rightarrow E(XY) = EX\ EY$

FALSE!!

$\boxed{\text{Conditional version. Expect } E(XY \mid \mathcal{F}) = E(X \mid \mathcal{F})\, E(Y \mid \mathcal{F})}$

To show $[M,N] = 0$, only NTS MN is a mg.

$$E\big(M(t)N(t) \mid \mathcal{F}_s\big) \overset{\text{ind}}{=\!=} E\big(M(t) \mid \mathcal{F}_s\big)\, E\big(N(t) \mid \mathcal{F}_s\big) = M(s)N(s)$$

$\Rightarrow MN$ is a mg $\Rightarrow$ QED.

Correct Proof: Let P be a partition of $[0,T]$.

$$E\left(\sum_{i=0}^{n-1}(\Delta_i M)(\Delta_i N)\right)^2 = E\sum_{i=0}^{n-1}(\Delta_i M)^2(\Delta_i N)^2$$

$$+\,2\,E\sum_{j=0}^{n-1}\sum_{i=0}^{j-1}(\Delta_i M)(\Delta_i N)(\Delta_j M)(\Delta_j N)$$

Note $E\big((\Delta_i M)(\Delta_i N)\,(\Delta_j M)(\Delta_j N)\big) \overset{\text{ind}}{=} \big(E(\Delta_i M)(\Delta_j M)\big)\cdot\big(E(\Delta_i N)(\Delta_j N)\big)$

$$= E\Big(E\big(\Delta_i M\,\Delta_j M \mid \mathcal{F}_{t_j}\big)\Big)\cdot E\big(\quad\big)$$

$$= \quad 0 \quad\cdot\quad E \quad = 0.$$

$$\Rightarrow E\left(\sum (\Delta_i M)(\Delta_i N)\right)^2 = E\sum (\Delta_i M)^2 (\Delta_i N)^2$$

$$\overset{\text{ind}}{=} \sum E(\Delta_i M)^2 \; E(\Delta_i N)^2$$

$$\le \underbrace{\left(\max_i E(\Delta_i M)^2\right)}_{\substack{\downarrow\, \|P\|\to 0 \\ 0}} \cdot \underbrace{E\sum (\Delta_i N)^2}_{\substack{\downarrow\, \|P\|\to 0 \\ E[N,N](T).}}$$

(by continuity).

$$\Rightarrow E[M,N]^2 = \lim E\left(\sum (\Delta_i M)(\Delta_i N)\right)^2 \overset{\text{(by above)}}{=} 0 \Rightarrow [M,N]=0$$

QED.

Remark: $[M,N]=0 \implies M$ & N are ind.

Eg: $M(t) = \int_0^t \mathbb{1}_{\{W(s)<0\}} \, dW(s)$.

$N(t) = \int_0^t \mathbb{1}_{\{W(s)\geq 0\}} \, dW(s)$.

[M]. M, N are mg. $[M,N] = \int_0^t 0 \, ds = 0$

(M & N not ind.

(Needs checking, $M+N=W$, -- You check ind).

Def (~~High~~ Multi Dim B. M.) .

We say a process $W = (W_1, W_2, \cdots W_d)$ is a standard d- dimensional B. M. if

① Each $W_i(t)$ is a standard ~~1~~ BM.

& ② For $i \neq j$, W_i & W_j are independent.

Note: If $i \neq j$, $[W_i, W_j] = 0$

If $i = j$ $[W_i, W_j](t) = t$

$$\left.\begin{array}{l} \text{If } i \neq j,\ [W_i, W_j] = 0 \\ \text{If } i = j\ \ [W_i, W_j](t) = t \end{array}\right\} [W_i, W_j](t) = 1_{\{i=j\}} t$$

or $d[W_i, W_j](t) = 1_{\{i=j\}} dt$

Theorem (Levy): If $M = (M_1, M_2, \cdots M_d)$ is a ds mg such that $M(0) = 0$ & $d[M_i, M_j](t) = 1_{\{i=j\}} \, dt$ then M is a ~~d~~ standard d-dim B.M.

$W \longrightarrow$ 2D ~~B~~ B.M.

~~$f(W)$~~ $d(f(t, W(t))) = \partial_t f \, dt + \partial_1 f \, dW_1 + \partial_2 f \, \partial W_2$

$$+ \frac{1}{2}\left(\partial_1^2 f \, dt + 2\partial_1 \partial_2 f \cdot 0 + \partial_2^2 f \, dt \right)$$

$$= \partial_t f \, dt + \partial_1 f \, dW_1 + \partial_2 f \, dW_2 + \frac{1}{2} \Delta f \, dt$$

$$\Delta f = \partial_1^2 f + \partial_2^2 f \qquad \text{(in 2D)}.$$

(Called the Laplacian).

Eg: $f(x) = \ln(x_1^2 + x_2^2)$.

You compute $\Delta f = 0$

$$f(W(t)) = Y(t)$$

$$dY(t) = \partial_1 f \, dW_1(t) + \partial_2 f \, dW_2(t) + 0 \, dt$$

$\Rightarrow Y$ is a mg

(But I claim Y is NOT a mg. You find the mistake (IOU on Wed).