Rec 5

Today: Extensions to Black-Scholes.

1) BS with Dividends.

(i.e. suppose S pays dividends).

Why do dividends even matter for option pricing?.

Intuitively, we price by hedging/replicating.

So we take positions in S, which pays dividends, but the option does not pay dividends.

ASSumptions:

~~Not~~ Non-Dividend.

BS assumptions.

1. S follows GBM i.e. $dS_t = \mu S dt + \sigma S dW_t$.
2. Access to risk free Bank Account.

$$dB = rB dt.$$

3. Can short/buy fractions of stocks.
4. Frictionless Market.
5 No arbitrage in the market

Modelling Dividends. (continuously).

6). $dD = \delta S dt$.

Reality: 1) Dividends paid out discretely (e.g. quarterly).

2). We don't know dividend schedules, so we estimate them anyway.

3). Generally companies like to pay more dividends when stock is doing well.

4). We assume dividends are reinvested in S.

Let's derive a PDE for a European option:

$V(S,t)$ is the value of some Europion option written on S and expiring at time T.

$$\pi = V - \Delta S.$$

if $\Delta < 0$. I own stock so I receive dividends.

i.e. $-q\Delta S dt$ "between time t, $t+dt$"

if $\Delta > 0$ I am short, so I have to pay dividends through to the owner of the stock.

i.e. I "get" $-q\Delta S dt$ (paying out).

$$\partial \pi = \partial V - \Delta \partial S - q\Delta S dt.$$

$$\overset{\text{Ito}}{=} \left(\frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt\right) - \Delta dS - q\Delta S dt$$

$$= \left(\frac{\partial V}{\partial S} - \Delta\right) dS + \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \delta \Delta S\right) dt$$

choose $\Delta = \frac{\partial V}{\partial S}$, In this case:

because only "dt" terms

$$\partial \pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \delta \frac{\partial V}{\partial S} S\right) dt \stackrel{\swarrow}{=} r\pi\, dt = r\left(V - \frac{\partial V}{\partial S} S\right) dt$$

$$\Rightarrow \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV = 0.$$

This is the BS PDE with dividends.

Before we continue with options let's consider a "simpler" case of a prepaid forward.

i.e. I pay now for an asset S, but I receive the asset at time T.

First let's do the case where S pays no dividends.

Intuitively clear it should be S_t (or S_0 at time 0).

Notice also a prepaid forward is a call with $K=0$.

If you use BS formula (overkill) you will see at $K=0$ you will get S_t.

BS price $C(S,t) = S\,N(d_1) - Ke^{-r(T-t)}N(d_2)$.

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t}.$$

so $K \to 0 \Rightarrow C(S,t) = S_t$

so what is $\Delta := \frac{\partial V}{\partial S} = 1$. always.

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so what about if S pays cont. dividends at rate q?

If I buy 1 share at time t, because of dividends at time T I will have accumulate $e^{q(T-t)}$ shares.

So if $\Delta = 1$ as before and I "hedge" by creating the portfolio $\Pi = V - S$.

Then at time T I have $S - Se^{q(T-t)}$ so I'm not hedged!.

So if instead we take $\Delta = e^{-q(T-t)}$, this will accumulate to 1 share at time T. called a tailed position.

because we get $\Delta S e^{q(T-t)}$ at time T. and we will be hedged.

Back to options pricing with dividends.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0. \quad (*)$$

suppose W is the exact same option written on the "same" stock, that pays no dividends

I.e. W satisfies regular BS.

Then $V(S,t) = W(Se^{-q(T-t)}, t)$, V will satisfy $(*)$

Let's Check.

$$\frac{\partial V}{\partial t}(S,t) = \frac{\partial W}{\partial S}\overset{\text{mult}}{*}(q, S e^{-q(T-t)}) + \frac{\partial W}{\partial t} \quad \text{evaluated at } (S e^{-q(T-t)}, T)$$

$$\frac{\partial V}{\partial S}(S,t) = \frac{\partial W}{\partial S}(S e^{-q(T-t)}, t)\, e^{-q(T-t)}$$

$$\frac{\partial^2 V}{\partial S}(S,t) = \frac{\partial^2 W}{\partial S^2}(S e^{-q(T-t)}, t)\, e^{-2q(T-t)}$$

From ⊛ we have: $\frac{\partial W}{\partial S}((S e^{-q(T-t)}, t)\, q S e^{-q(T-t)} + \frac{\partial W}{\partial t}(S e^{-q(T-t)}, t)$

$$+ \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial S^2}(S e^{-q(T-t)}, t) e^{-2q(T-t)} + (r-q) S \frac{\partial W}{\partial S}(S e^{-q(T-t)}, t) e^{-q(T-t)}$$

$$- r W(S e^{-q(T-t)}, t)$$

$$= \frac{\partial W}{\partial t}(S e^{-q(T-t)}, t) + \frac{1}{2}\sigma^2 (S e^{-q(T-t)})^2 \frac{\partial^2 W}{\partial S^2}(S e^{-q(T-t)}, t)$$

$$+ r S e^{-q(T-t)} \frac{\partial W}{\partial S}(S e^{-q(T-t)}, t) - r W(S e^{-q(T-t)}, t) = 0$$

set $R = S e^{-q(T-t)}$.

$$\frac{\partial W}{\partial t}(P,t) + \frac{1}{2}\sigma^2 P^2 \frac{\partial^2 W}{\partial P^2}(P,t) + rP\frac{\partial W}{\partial P}(P,t) - rW(P,t) = 0.$$

If we want to price the call we have ⊗ as the PDE with BC's ~~H~~ FC

$$C(S,T) = (S-K)^+ \quad (FC).$$

$$C(0,T) = 0.$$

$$\lim_{S\to\infty} \frac{\partial C}{\partial S}(S,t) = e^{-q(T-t)}$$

for S big. $C \approx S-K \approx S$.

$S \gg K$.

Now by our formula!

$$C(S,T) = C_{BS}(Se^{-q(T-t)}, t).$$

$$= Se^{-q(T-t)}N(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$\text{where } d_1 = \frac{\ln\left(\frac{Se^{-q(T-t)}}{K}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

$$\ln\left(\frac{Se^{-q(T-t)}}{K}\right) = \ln\left(\frac{S}{K}\right) + \ln\left(e^{-q(T-t)}\right) = -q(T-t).$$

This is often written in this way...

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2) How do we price options on more than one underlying?

suppose
$$\begin{cases} \partial S = \mu_1 S dt + \sigma_1 S dW_t \\ \partial R = \mu_2 R dt + \sigma_2 R \partial B_t \\ E[\partial W_t \partial B_t] = \rho\, dt. \end{cases}$$
B, W are different Brownian motions.

Let $V(S, R, t)$ be a European option written on R, S

(ex. Margrabe option which allows the owner the right, but not obligation to receive a units of S for b units of R

i.e. $V(S, R, T) = \max(aS - bR, 0)$.

Let's derive our BS PDE in this case.
(S here does not pay dividends).

$$\pi = V - \Delta_S S - \Delta_R R.$$

Apply 2D Ito:

$$\partial\pi = \partial V - \Delta_S \partial S - \Delta_R \partial R.$$

$$= \Big(\frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{\partial V}{\partial R}dR + \frac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 V}{\partial S^2}dt$$

$$+ \frac{1}{2}\sigma_2^2 R^2 \frac{\partial^2 V}{\partial R^2}dt + \rho\sigma_1\sigma_2 RS \frac{\partial V}{\partial S}\frac{\partial R}{\partial R}dt\Big)$$

$$- \Delta_S dS - \Delta_R \partial R.$$

set $\Delta_S = \frac{\partial V}{\partial S}$, $\Delta_R = \frac{\partial V}{\partial R}$ we will lose the stochastic terms:

$$\partial\pi = \Big(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho\sigma_1\sigma_2 RS \frac{\partial V}{\partial S}\frac{\partial V}{\partial R} + \frac{1}{2}\sigma_2^2 R^2 \frac{\partial^2 V}{\partial R^2}\Big)dt$$

$$= r\pi\, dt = r\Big(V - \frac{\partial V}{\partial S}S - \frac{\partial V}{\partial R}R\Big)dt.$$

$$\Rightarrow \frac{\partial U}{\partial t} + \frac{1}{2}\sigma_1^2 S^2 \frac{\partial^2 U}{\partial S^2} + \rho\,\sigma_1\sigma_2\, RS \frac{\partial U}{\partial S}\frac{\partial U}{\partial R} + \frac{1}{2}\sigma_2^2 R^2 \frac{\partial^2 U}{\partial R^2}$$

$$+ rS\frac{\partial U}{\partial S} + rR\frac{\partial U}{\partial R} - rU = 0.$$

Once again this is the "global rule" for European options, contract specifications appear as BC's + FC

For The Magrabe option:

$$V(S,R,T) = \max(aS - bR, 0) \quad (FC)$$

BC's: $V(0,R,t) = 0$, $V(S,0,t) = aS$.

$\lim_{S\to\infty} \frac{\partial V}{\partial S}(S,R,t) = a$, $\lim_{R\to\infty} V(S,R,t) = 0$

sol'n to this problem:

$aS\,N(d_1) - bR\,N(d_2)$ where:

$$d_1 = \frac{\ln\left(\frac{S}{R}\right) + \left(r + \frac{1}{2}\sigma_x^2\right)(T-t)}{\sigma_x\sqrt{T-t}}, \quad d_2 = d_1 - \sigma_x\sqrt{T-t}$$

$$\sigma_x^2 := \sigma_1^2 - \rho\sigma_1\sigma_2 + \sigma_2^2$$