Black - Scholes - Merton.

1. Model stock price by ~~GMM~~ GBM (α, σ).

$$dS(t) = \alpha \cancel{d} S\,dt + \sigma S\,dW$$

$\alpha \Rightarrow$ mean return

$\sigma \Rightarrow$ % volatility

2. MM with interest rate r

3. European call on the stock, strike K, maturity T.

$$\overset{c(t,x)}{\cancel{c(x,t)}} = \text{AFP of the call } \cancel{\text{it}} \text{ if } S(t) = x$$

$c \Rightarrow$ non random.

Last time:

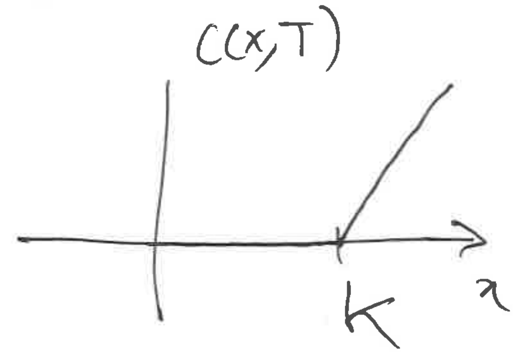
① If $c(t,x)$ = AFP of call given $S(t) = x$.

then c satisfies the BSM PDE

ⓐ $\partial_t c + r x \partial_x c + \frac{\sigma^2 x^2}{2} \partial_x^2 c = r c$

ⓑ $c(t, 0) = 0$

& ⓒ $c(T, x) = (x-K)^+$



& ② Conversely if c satisfies ⓐ, ⓑ & ⓒ, then

$c(t, S(t))$ = AFP of the call.

Redo Proof of 2.

Construct a R- Portfolio.

Choose 1 $X(0) = c(0, S(0))$.

2 Hold $\Delta(t)$ shares of the stock & rest in M.M.

Choose $\Delta(t) = \partial_x c(t, S(t))$.

Claim: X is a R. portfolio.

<u>Claim</u>: $X(t) = c(t, S(t))$ for all $t < T$

Last time: computed $d\left(e^{-rt}X(t)\right)$ & $d\left(e^{-rt}c(t,\overset{S}{\cancel{X}}(t))\right)$.

& saw they were equal.

Redo without discounting:

Let $Y(t) = X(t) - c(t, S(t))$.

Compute dY: ① $dX = \Delta(t)\,dS(t) + \big(X(t) - \Delta(t)S(t)\big)\,r\,dt$

② $d\big(c(t,S(t))\big) = dc = \partial_t c\,dt + \partial_x c\,dS + \frac{1}{2}\partial_x^2 c\,d[S,S]$.

$$= \partial_t c\,dt + \partial_x c\,dS + \frac{1}{2}\sigma^2 S^2 \cancel{dt}\,\partial_x^2 c\,dt$$

$$\Rightarrow dY = \underbrace{\big(\Delta(t) - \partial_x c\big)}_{=0}\,dS - \Big(\partial_t c + \Delta(t)S(t)\cdot r - Xr + \frac{1}{2}\partial_x^2 c\,\sigma^2 S^2\Big)\,dt$$

$$\Rightarrow dY = 0 - \left(\partial_t c + rS(t)\,\partial_x c + \frac{\sigma^2}{2} S^2 \partial_x^2 c - rX\right) dt$$

$$\overset{(PDE)}{=} -\left(rc - rX\right) dt$$

$$= r(X - c)\, dt \quad = rY\, dt$$

$$\Rightarrow Y(t) = Y(0)\, e^{rt} = \underbrace{\left(X(0) - \overset{c(0,S(0))}{\cancel{c(t, S(t))}}\right)}_{0} \cdot e^{rt}$$

$$\Rightarrow Y(t) = 0 \Rightarrow X(t) = c(t, S(t)). \quad \text{for all } t < T$$

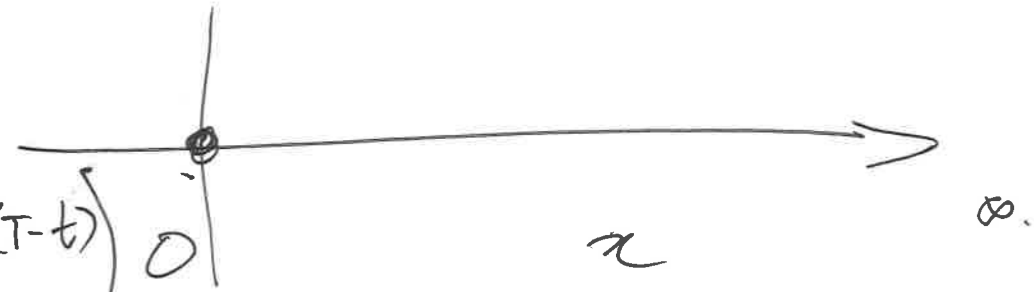
By continuity $X(T) = c(T, S(T)) = (S(T) - K)^+$

$\Rightarrow X$ is a R-portfolio $\Rightarrow X(t) = AFP. \Rightarrow c(t, S(t)) = AFP$

QED.

B.C. at $x = \infty$ is

$$\lim_{x \to \infty} \left(c(t,x) - (x-K)e^{-r(T-t)} \right) = 0$$



Price Put: $p(t,x)$ = European put, strike K,
mat T
given $S(t) = x$.

Put call parity: Know $p(T,x) = (K-x)^+$

Compute $c(T,x) - p(T,x) = (x-K)^+ - (K-x)^+ = x - K$.

Consider a Pf that is long 1 call & short 1 put.

Payoff $= S(T) - K$ (Forward contract).

Replicate by buying ~~&~~ the stock & discounting K in cash.

Value at time $t = S(t) - Ke^{-r(T-t)}$

$$\Rightarrow \cancel{c(t)}\, c(t, S(t)) - p(t, S(t)) = S(t) - Ke^{-r(T-t)}$$

$$\boxed{\Rightarrow p(t, S(t)) = c(t, S(t)) - S(t) + Ke^{-r(T-t)}}$$

Greeks: Delta $\longrightarrow$ $\partial_x c$

Recall: $c(t,x) = x N(d_+) - K e^{-r(T-t)} N(d_-)$

$$d_\pm = \frac{1}{\sigma\sqrt{(T-t)}}\left(\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t) \right)$$

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \, dy$$

Delta: $\partial_x c = N(d_+) + x N'(d_+)\partial_x d_+ - K e^{-r(T-t)} N'(d_-)\partial_x d_-$

turns out $x N'(d_+) d_+' = K e^{-r(T-t)} N'(d_-) d_-'$

$$\Rightarrow \partial_x c = N(d_+)$$

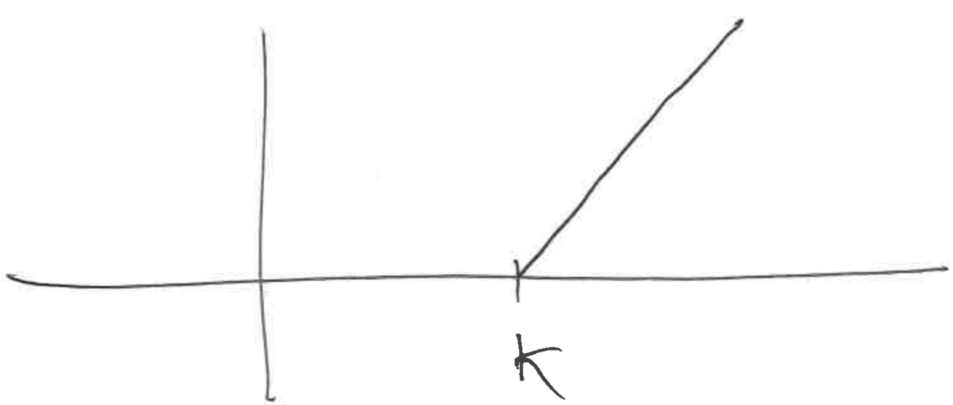
$$\Rightarrow \partial_x c = N(d_+) > 0$$

$$\boxed{\Rightarrow c \text{ is increasing as a fn of } x.}$$

(2) Gamma: $\partial_x^2 c$ (You compute)

$$\partial_x^2 c = \frac{1}{x\sigma\sqrt{2\pi(T-t)}} e^{-(d_+^2/2)}.$$

$\partial_x^2 c > 0 \Rightarrow c$ is convex as a fn of x.

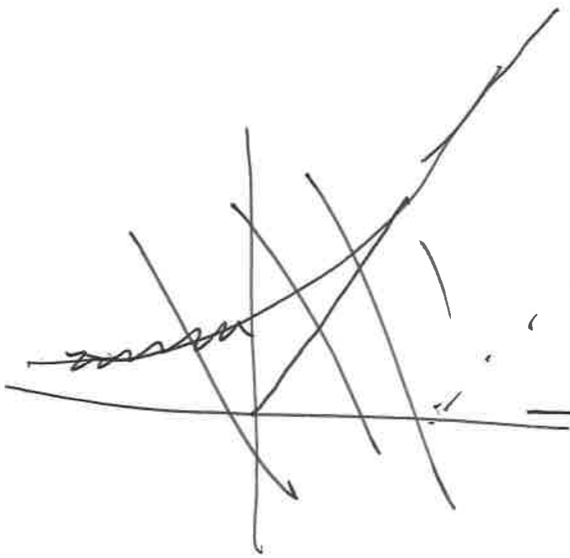


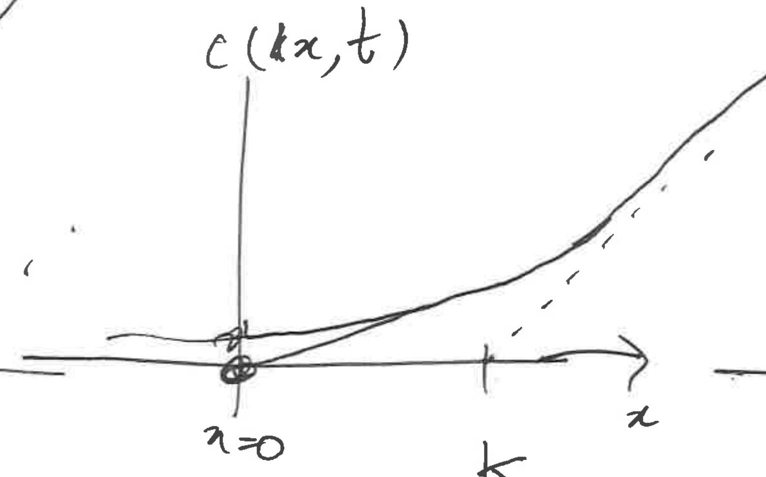
$t = T$

③ Theta: $\partial_t c = $ (You check) $=$

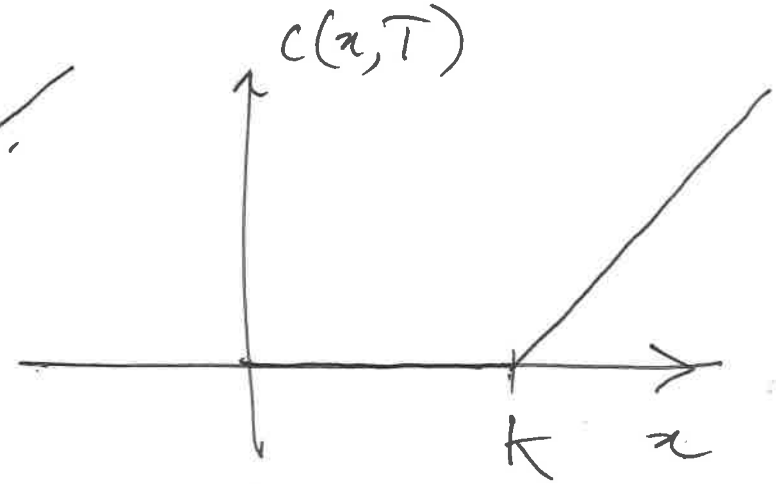
$$= -rKe^{-r(T-t)}N(d_-) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+)$$

$\Rightarrow \partial_t c < 0 \Rightarrow c$ is decreasing as a fn of t.





Some $t < T$



$t = T$

Hedging a short call:

Sell a call option: $c(t,x)$.

Hedge this:

R- portfolio holds $\partial_x c(t,x)$ in the asset

& rest in cash.

Cash value in MM $= c(t,x) - x\,\partial_x c(t,x)$

~~#~~ Compute using formulae for c

$$= x N(d_+) - K e^{-r(T-t)} N(d_-) - x N(d_+).$$

$$= -K e^{-r(T-t)} N(d_-) < 0$$

Multi-Dimensional Itô Calculus.

X, Y two processes. (cts & adapted).

<u>Def</u>: The Joint Quadratic Variation of X & Y denoted by $[X, Y](t)$

$$[X,Y](t) = \lim_{\|P\| \to 0} \sum \left(X_{t_{i+1}} - X_{t_i}\right)\left(Y_{t_{i+1}} - Y_{t_i}\right)$$

(P = partition of $[0, t]$
$= \{0 = t_0 < t_1 < t_2, \dots t_n = t\}$.

Note $4ab = (a+b)^2 - (a-b)^2$

$\Delta_i X = X_{t_{i+1}} - X_{t_i}$ & $\Delta_i Y = Y_{t_{i+1}} - Y_{t_i}$

$$\sum (\Delta_i X)(\Delta_i Y) = \frac{1}{4} \sum \left[\left(\Delta_i (X+Y)\right)^2 - \left(\Delta_i (X-Y)\right)^2 \right]$$

$$\xrightarrow{\|P\| \to 0} \frac{1}{4} \Big([X+Y, X+Y] - [X-Y, X-Y] \Big).$$

Hence $[X, Y] = \frac{1}{4} \Big([X+Y, X+Y] - [X-Y, X-Y] \Big).$

Prop: (Product rule)

If X & Y are two processes, then

$$d(XY) = X\,dY + Y\,dX + d[x,y].$$

Pf:

$$d(X+Y)^2 = 2(X+Y)\,d(X+Y) + d[X+Y, X+Y].$$

$$d(X-Y)^2 = 2(X-Y)\,d(X-Y) + d[X-Y, X-Y].$$

$$\Rightarrow d\left((X+Y)^2 - (X-Y)^2\right) = 4(X\,dY + Y\,dX) + 4\,d[X,Y].$$

$$\Rightarrow 4\,d(XY) = 4X\,dY + 4Y\,dX + 4\,d[x,y].$$

QED.

~~Prop:~~ $\boxed{d[X,Y] = dX\,dY}$ ← Notation.

Prop: If X is an Ito process & B is cts, finite 1^{st} variation.

then $[X,B] = 0$

Pf: $[X,B] = \frac{1}{4}\Big([X+B, X+B] - [X-B, X-B]\Big)$.

$= \frac{1}{4}\big([X,X] - [X,X]\big) = 0$.

Multi D Ito formula:

Say $X_1, X_2, \cdots X_n$ are n Itô processes.

Let $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is C^1 wrt to t.

t

& C^2 wrt $x_1, \cdots x_n$.

$(f = f(t, x_1, x_2, \cdots x_n)).$

$\left(\partial_t f, \frac{\partial f}{\partial x_i} \text{ \& } \frac{\partial^2 f}{\partial x_i \partial x_j}\right.$ exist & are cts.$\left.\right)$.

Set $Y(t) = f(t, X_1(t), X_2(t), \cdots X_n(t))$.

$= f(t, X(t))$ (where $X(t) = (X_1(t), X_2(t) \cdots X_n(t))$.

Multi D Itô says.

$$dY = \partial_t f(t, X(t))\, dt + \sum_{i=1}^{n} \partial_i f(t, X(t))\, dX_i(t)$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} \partial_i \partial_j f(t, X(t)) \cdot d[X_i, X_j](t).$$