Today — Black Scholes Merton formula.

Setup: ① Asset whose price is $S(t)$ & constant return rate α.

$$\left.\begin{aligned} & S(t) = S(0)e^{\alpha t} \\ \iff & dS(t) = \alpha S(t)\,dt \end{aligned}\right\}$$

(M.M.).

② Stock prices: Model ~~by~~ price by

$$dS(t) = \alpha S(t)\,dt + \sigma S(t)\,dW(t).$$

$\alpha \to$ mean return rate

$\sigma \to$ % volatility

Def: A geometric BM with param α & σ is defined to be a process S such that

$$dS(t) = \alpha S\, dt + \sigma S\, dW(t)$$

Remk: Let $Y(t) = \ln S(t)$.

$$\boxed{\begin{aligned} &f(x) = \ln x, \quad \partial_t f = 0 \\ &\partial_x f = \frac{1}{x}, \quad \partial_x^2 f = -\frac{1}{x^2} \end{aligned}}$$

Ito:

$$dY = \partial_t f\, dt + \partial_x f\, dS + \frac{1}{2}\partial_x^2 f\, d[S,S].$$

$$= 0 + \frac{1}{S}\left(\alpha S\, dt + \sigma S\, dW\right) - \frac{1}{2S^2}\sigma^2 S^2 dt$$

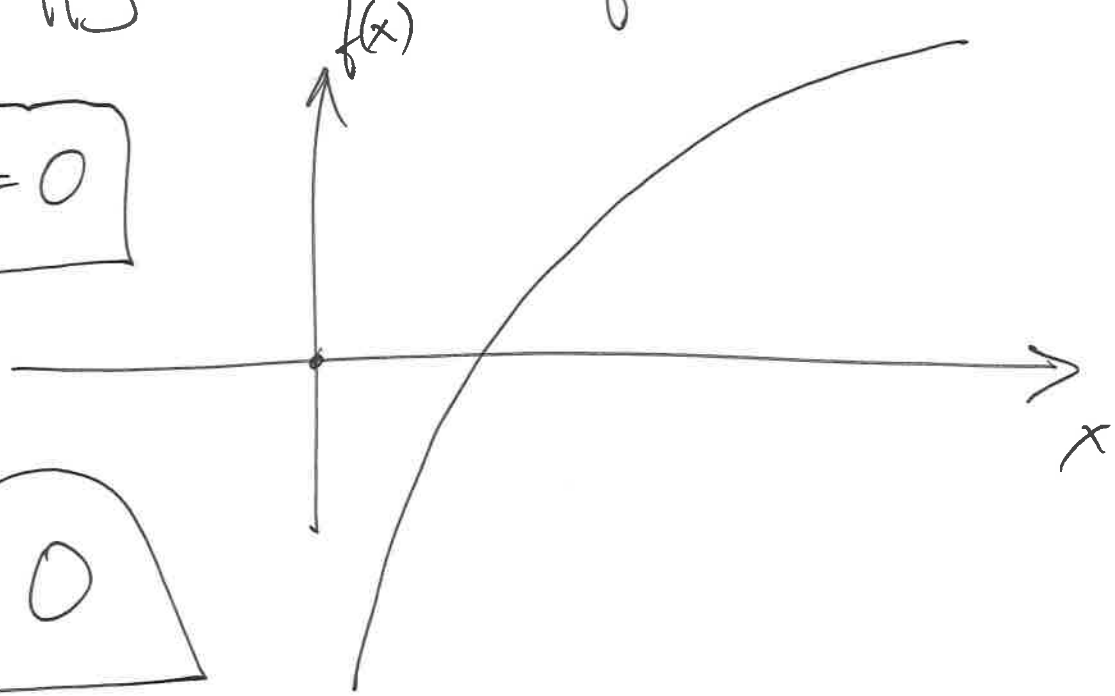
$$= \left(\alpha - \frac{\sigma^2}{2}\right) dt + \sigma\, dW$$

$$\Rightarrow \underbrace{Y(t) - Y(0)}_{\ln\left(\frac{S(t)}{S(0)}\right)} = \left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W(t)$$

$$\Rightarrow S(t) = S(0) \cdot \exp\left(\left(\alpha - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right).$$

Note: To apply Itô need $f \in C^{1,2}$

$\boxed{\text{not } C^2 \text{ at } x = 0}$



OK because $S(t)$ never hits 0

European call option: Strike K, Maturity T.

Goal: Find the arbitrage free price of this option.

~~Theorem:~~ Remk: Black & Scholes: AFP at time t

only depends on $S(t)$ (& parameters K, T, α, σ, r)

interest rate.

Thm: Say we have an arbitrage free market

consisting of (1) MM account with return rate r

& (2) Risky asset (stock) modeled by $GBM(\alpha, \sigma)$

consider Europen call, strike K & maturity T

① If $c = c(t,x)$ ~~$c(x,t)$~~ such that at any time $t \leq T$ the AFP of option is $c(t, S(t))$

then c satisfies.

called the B-S-M PDE

$$\begin{cases} \text{(a)} \quad \partial_t c + r x \partial_x c + \dfrac{\sigma^2 x^2}{2} \partial_x^2 c - rc = 0 \\ \text{(b)} \quad c(t, 0) = 0 \\ \text{(c)} \quad c(T, x) = (x - K)^+ = \max\{x-K, 0\}. \end{cases}$$

Thm ② Conversely if c satisfies ⓐ, ⓑ & ⓒ then $c(t, S(t)) = $ AFP of call at time t.

Rmk: Soln to ⓐ, ⓑ, ⓒ is explicitly given by.

$$c(t,x) = x\, N\big(d_+(T-t, x)\big) - K e^{-r(T-t)} N\big(d_-(T-t,x)\big).$$

where

$$d_\pm = \frac{1}{\sqrt{\sigma^2(T-t)}}\left(\ln\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)\right).$$

& $N(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-y^2/2}\, dy$

Assumptions: ① Frictionless market

② liquidity

③ Borrow & lend from MM, interest rate r.

Idea — Replicating portfolio.

$$X(t) \begin{cases} \to \Delta(t) \text{ shares of } \cancel{\$} \text{ the stock.} \\ \to \text{Rest in MM (return rate } r\text{).} \end{cases}$$

Want to choose $\Delta(t)$ so that

$$X(T) = (S(T)-K)^+$$

(X has the same ~~as~~ cash flow as the call option).

arbitrage free pricing dictates: $X(t) = \text{AFP}$ of call option at time t.

<u>Remark:</u> (Delta Hedging rule). we will see that

$$\Delta(t) = \partial_x c(t, S(t)).$$

Proof of BS part 1:

Let $X(t) =$ value of ~~rep~~ R. portfolio at time t.

(X holds $\Delta(t)$ shares of S & rest in MM),

Know $X(t) = \text{AFP}$ of call.

For part 1, we assume AFP of call is $c(t, S(t))$.

Pf. $\Rightarrow X(t) = c(t, S(t))$.

Need to show c satisfies the BSM PDE (a, b, c).

1. Compute dX:

$$dX = \Delta(t)\, dS(t) + (X(t) - \Delta(t) S(t))\, r\, dt \quad \cdots \text{(*)}$$

2. Compute $d\, c(t, S(t))$ by Itô & equate to

Know $d\, c(t, S(t)) = dc$

$$= \partial_t c\, dt + \partial_x c\, dS + \frac{1}{2} \partial_x^2 c\, d[S, S].$$

recall $dS = \alpha S\,dt + \sigma S\,dW \Rightarrow d[S,S](t) = \sigma^2 S^2 dt$

$$\Rightarrow dc = \partial_t c\,dt + \partial_x c\left(\alpha S\,dt + \sigma S\,dW\right) + \frac{\partial_x^2 c}{2}\sigma^2 S^2 dt$$

$$= \left(\partial_t c + \alpha S \partial_x c + \frac{\sigma^2 S^2}{2}\partial_x^2 c\right)dt + \sigma S \partial_x c\,dW$$

equate to (*)

$$= \Delta(t)\left(\alpha S\,dt + \sigma S\,dW\right) + \left(X - \Delta S\right) r\,dt$$

$$= \left(\alpha S \Delta + (X - \Delta S) r\right)dt + \sigma S \Delta\,dW$$

$\Rightarrow$ By uniqueness of the Itô decomposition

$$\Rightarrow \sigma S\, \partial_x c(t, S(t)) = \sigma S\, \Delta(t) \quad \text{--} \quad ①$$

$$\& \quad \partial_t c + \alpha S\, \partial_x c + \frac{\sigma^2 S^2}{2} \partial_x^2 c = \alpha S \Delta + (X - \Delta S) r \quad \ldots ②$$

① $\Rightarrow$ Delta Hedging : $\boxed{\partial_x c(t, S(t)) = \Delta(t)}$

$$② \Rightarrow \partial_t c(t, S(t)) + \cancel{\alpha S(t)\, \partial_x c(t, S(t))} + \frac{\sigma^2 S(t)^2}{2} \partial_x^2 c(t, S(t))$$

$$= \cancel{\alpha S(t)\, \Delta(t)} + (X - \Delta(t) S(t)) r$$

$$\Rightarrow \partial_t c(t, S(t)) + r S(t)\, \partial_x c(t, S(t)) + \frac{\sigma^2 S(t)^2}{2} \partial_x^2 c(t, S(t))$$

$$- r \underbrace{X}_{= c(t, S(t))} = 0$$

Replace $X(t)$ with $c(t, S(t))$

& $S(t)$ with x & get.

$$\partial_t c(t,x) + r x \, \partial_x c(t,x) + \frac{\sigma^2}{2} x^2 \partial_x^2 c(t,x) - r x = 0$$

QED.

Part ② of theorem: Now assume $c(t,x)$ satisfies the BSM PDE.

Want to show that $c(t, S(t)) = \text{AFP}$ of the call.

<u>Idea:</u> Construct a R. Portfolio.

Let $X(t)$ be the value of a pf that holds. $\Delta(t)$ shares of S & rest in M.M.

Choose $\Delta(t) = \partial_x c(t, S(t))$.

Goal: Show $X(T) = (S(T) - K)^+$

If we show $X(T) = (S(T) - K)^+$

$\Rightarrow X(t) = AFP.$

will show $X(t) = c(t, S(t)) \Rightarrow c(t, S(t)) = AFP \Rightarrow QED.$

<u>Trick 2:</u> $Y(t) = e^{-rt} X(t)$.

$$\boxed{\begin{aligned} dX &= \Delta \, dS + (X - \Delta S) \, r \, dt \\ dS &= \alpha S \, dt + \sigma S \, dW \end{aligned}}$$

ⓐ Compute dY: $dY = -r e^{-rt} X(t)\, dt + e^{-rt} dX(t)$

$$dY = e^{-rt} (\alpha - r) \Delta(t) S(t)\, dt + e^{-rt} \sigma \Delta(t) S(t)\, dW(t).$$

ⓑ Compute $d\left(e^{-rt} c(t, S(t))\right)$ using Itô.

$$d\left(e^{-rt} c(t, S(t))\right) = e^{-rt} (\alpha - r) S \, \partial_x c \, dt + e^{-rt} \partial_x c \, \sigma S \, dW$$

$$= dY \quad \left(\text{since } \partial_x c(t, S(t)) = \Delta(t) \text{ by choice}\right).$$

$$= d\left(e^{-rt} X(t)\right).$$

$$\Rightarrow e^{-rt}X(t) = X(0) + \int_0^t dY$$

$$= X(0) + \int_0^t d\left(e^{-rt}c(t,S(t))\right)$$

$$= X(0) + e^{-rt}c(t,S(t)) - c(0,S(0)).$$

$$\Rightarrow e^{-rt}X(t) = e^{-rt}c(t,S(t)) + \underbrace{X(0) - c(0,S(0))}_{\cancel{\mathrm{IOU} = 0}}.$$

$$\Rightarrow X(t) = c(t,S(t)).$$

QED.