$\boxed{7.5}$ (From notes).

$\sigma \longrightarrow$ Not random.

$$X(t) = \int_0^t \sigma(u)\, dW(u)$$

Last time: $$E\left(e^{\lambda(X(t)-X(s))} \,\middle|\, \mathcal{F}_s\right) = e^{\frac{\lambda^2}{2}\int_s^t \sigma(u)^2\, du}$$

$$\Rightarrow E\left(e^{\lambda(X(t)-X(s))}\right) = E$$

$$= e^{\lambda^2/2 \int_s^t \sigma^2(u)\, du}$$

$$= \text{MGF}\left(N\left(0, \int_s^t \sigma^2(u)\, du\right)\right).$$

(f) Compute $E\left(e^{\lambda X(r) + \mu(X(t)-X(s))}\right) = \text{MGF}\left(X(r), X(t)-X(s)\right).$

$$= e^{\frac{\lambda^2}{2}\int_0^r \sigma^2(u)\,du + \frac{\mu^2}{2}\int_s^t \sigma^2(u)\,du} \longleftarrow \text{MGF of Joint normal.}$$

$$= \left(E\, e^{\lambda X(r)}\right)\left(E\, e^{\mu(X(t)-X(s))}\right).$$

(c) $\Rightarrow$ (1) $X(t)-X(s)$ ind of $X(r)$

& (2) $\left(X(r),\ X(t)-X(s)\right) \sim N\left(0, \begin{pmatrix} \int_0^r \sigma^2(u)\,du & 0 \\ 0 & \int_s^t \sigma^2(u)\,du \end{pmatrix}\right).$

(d) $\sigma^2 = 1$ (i.e. $\sigma = \pm 1$) $\Rightarrow X$ is a BM.

Check: $r < s < t$,

$$\cancel{X(t)} \; \big(X(r),\, X(t) - X(s)\big) \sim N\left(0, \begin{pmatrix} r & 0 \\ 0 & t-s \end{pmatrix}\right).$$

$\Rightarrow X(t) - X(s) \sim N(0, t-s)$

& ind of $X(r)$ for every $r \leq s$.

$\Rightarrow X(t) - X(s)$ ind of $\mathcal{F}_s = \sigma\left(\bigcup_{r \leq s} \sigma(X_r)\right)$

$\Rightarrow X$ is a BM.

Problem: Compute $\mathrm{Var}\left(\underbrace{\int_0^t \exp(-W_s^2)\, dW(s)}\right) = V.$

~~Know that~~ Let $X(t) = \int_0^t e^{-W(s)^2}\, dW(s).$

$$V = EX(t)^2 - \left(EX(t)\right)^2,$$

Know $EX(t) = 0$ $\left(\because X \text{ is a mg}\right.$

$$\left.\Rightarrow EX(t) = EX(0) = 0\right).$$

(Know $\int_0^t \sigma(s)\, dW(s)$ is a mg

Provided $E \int_0^t \sigma(s)^2\, ds < \infty$

Easy to see $E \int_0^t e^{-W(s)^2}\, ds < \infty \quad (< t).$

But $E \int_0^t e^{+W(s)^2}\, ds \underset{?}{<} \infty$ (suffer)?.)

$$E\, e^{W(t)^2} = \int_{-\infty}^{\infty} e^{x^2} \cdot \frac{1}{\sqrt{2\pi t}}\, e^{-x^2/2t}\, dx$$

$$= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left(-x^2\left(\frac{1}{2t} - 1\right)\right). dx$$

If $\frac{1}{2t} - 1 > 0$ (i.e. $t < \frac{1}{2}$).

$$E\left(e^{W(t)^2}\right) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-x^2(+ve)} \, dx < \infty.$$

But $\frac{1}{2t} - 1 \leq 0$

$$E\, e^{W(t)^2} = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} e^{-x^2(-ve)} \, dx = +\infty$$

Back to Problem: $EX(t) = 0$

$$EX(t)^2 = E\left(\int_0^t e^{-W(s)^2} dW(s)\right)^2.$$

(Itô isometry) $$= E\int_0^t \left(e^{-W(s)^2}\right)^2 ds.$$

$$= \int_0^t E e^{-2W(s)^2} ds.$$

$$= \int_0^t \int_{-\infty}^{\infty} e^{-2x^2} \frac{1}{\sqrt{2\pi s}} e^{-x^2/2s} dx\, ds,$$

Itô Isom: $E\left(\int_0^t \sigma(s)\, dW(s)\right)^2 = E\int_0^t \sigma(s)^2\, ds.$

$\boxed{\tau < t}$

Q: Is $E\left[\left(\int_0^t \sigma(s)\, dW(s)\right)^2 \Big| \mathcal{F}_\tau\right] \overset{?}{=} E\left(\int_0^t \sigma(s)^2 ds \Big| \mathcal{F}_\tau\right).$

Claim: False: Let $M(t) = \int_0^t \sigma(s)\, dW(s).$

Compute $E\left(M(t)^2 \big| \mathcal{F}_\tau\right).$

Know $d(M^2) = 2M\,dM + \frac{1}{2}\,2\,d[M,M]$.

$$d(M^2) = 2M\sigma\,dW + \sigma^2\,dt$$

$$\Rightarrow M(t)^2 - M(\tau)^2 = \int_\tau^t 2M(s)\sigma(s)\,dW(s) + \int_\tau^t \sigma^2(s)\,ds.$$

$$\Rightarrow E\left(M(t)^2 \mid \mathcal{F}_\tau\right) = M(\tau)^2 + 0 + E\left(\int_\tau^t \sigma(s)^2\,ds \mid \mathcal{F}_\tau\right).$$

$$\Rightarrow E\left(M(t)^2 \mid \mathcal{F}_\tau\right) = M(\tau)^2 - \int_0^\tau \sigma(s)^2\,ds + E\left(\int_0^t \sigma(s)^2\,ds \mid \mathcal{F}_\tau\right).$$

$$\Rightarrow E\left[\left(\int_0^t \sigma(s)\,dW(s)\right)^2 \Big| \mathcal{F}_\tau\right] = E\left(\int_0^t \sigma(s)^2\,ds \Big| \mathcal{F}_\tau\right)$$

$$\boxed{+ \; M(\tau)^2 - \int_0^\tau \sigma(s)^2\,ds.}$$

non zero!

$$\cancel{E\left(\left(M(t)^2 - M(\tau)^2\right)^2 \Big| \mathcal{F}_\tau\right) = X(t)}$$

$$M(t) = \int_0^t W(s)\,dW(s).$$

Compute $E\left(\left(M(t) - M(s)\right)^2 \,\middle|\, \mathcal{F}_s\right)$

$$= E\left(\left(\int_s^t W(\tau)\,dW(\tau)\right)^2 \,\middle|\, \mathcal{F}_\tau\right).$$

$$\Rightarrow E\left(M(t)^2 + M(s)^2 - 2M(t)M(s) \,\middle|\, \mathcal{F}_s\right).$$

$$= E\left(M(t)^2 \,\middle|\, \mathcal{F}_s\right) + M(s)^2 - 2M(s)^2$$

$$= E\left(M(t)^2 - M(s)^2 \,\middle|\, \mathcal{F}_s\right).$$

$M(t)^2 - [M,M](t)$ is a mg.

$\Rightarrow E\left(M(t)^2 - [M,M](t) \mid \mathcal{F}_s\right) = M(s)^2 - [M,M](s).$

$\Rightarrow E\left(M(t)^2 - M(s)^2 \mid \mathcal{F}_s\right) = E\left([M,M](t) - [M,M](s) \mid \mathcal{F}_s\right).$

$\Rightarrow E\left((M(t) - M(s))^2 \mid \mathcal{F}_s\right) = E\left(\int_s^t W(r)^2 dr \mid \mathcal{F}_s\right)$

$= \int_s^t E\left(W(r)^2 \mid \mathcal{F}_s\right) dr.$

& compute.

7.6: $X(t) = \int_0^t s \, dW(s)$

$Y(t) = \int_0^t W(s) \, ds$.

Compute $E X(t)^n$ & $E Y(t)^n$.

① $X(t) \sim N\left(0, \int_0^t s^2 ds\right) = N\left(0, \frac{t^3}{3}\right)$. (Q7.5).

Compute moments: ~~MGF~~ $\cancel{M_{X\lambda}(X(t)) = e^{\frac{\lambda^2}{2} \cdot t^3/3}}$

$M_{X(t)}(\lambda) = e^{\lambda^2/2 \cdot t^3/3}$.

$$E X^n = \frac{d^n}{d\lambda^n} M_{X(t)}(\lambda) \Big|_{\lambda = 0}$$

n^{th} term (n even)

$$\frac{\lambda^n}{(n/2)!}\left(\frac{t^3}{6}\right)^{n/2}$$

$$= \frac{d}{d\lambda^n}\left(1 + \frac{\lambda^2}{2}\frac{t^3}{3} + \frac{\left(\frac{\lambda^2}{2}\frac{t^3}{3}\right)^2}{2!} + \cdots\right)\Bigg|_{\lambda=0}$$

$$= \begin{cases} 0 & n \text{ is odd.} \\ \dfrac{n!}{(n/2)!}\left(\dfrac{t^3}{6}\right)^{n/2} & n \text{ is even.} \end{cases}$$

$Y(t) = \int_0^t W(s)\, ds$. Compute EY^n.

2 tricks: (1.) Try & ito something to get $\int_0^t W(s)\, ds$.

as one of the terms.

Trick (2) Realise $Y(t)$ is Normal.

$$Y(t) = \lim_{\|P\| \to 0} \sum W(t_i)(t_{i+1} - t_i).$$

$(W(t_0), W(t_1), \dots W(t_n))$ is Jointly Normal.

$\Rightarrow \sum W(t_i)(t_{i+1} - t_i)$ is also

$\Rightarrow \lim(\ )$ " "

$\Rightarrow$ $\int_0^t W(s)\,ds$ is Normal.

$\Rightarrow$ Can find $E\,Y^4_{(t)}$ in terms of $E\,Y^2_{(t)}$.

Need to compute $E\left(\int_0^t W(s)\,ds\right)^2 = \cancel{E\int_0^t W(s)^2\,ds}$

Note
$$E\left(\int_0^t W(s)\,ds\right)^2 = E\left(\int_0^t W(s)\,ds\right)\left(\int_0^t W(s)\,ds\right)$$
$$= E\left(\int_0^t W(s)\,ds\right)\left(\int_0^t W(r)\,dr\right).$$

$$= E\left(\int_0^t \int_0^t W(s)\,W(r)\, ds\, dr\right).$$

$$= \int_0^t \int_0^t E\,W(s)\,W(r)\; ds\, dr.$$

$$= \int_0^t \int_0^t (s \wedge r)\; ds\, dr,$$ & compute this!!

Q: $W \to$ BM.

Why is $\left(W(t),\ W(s) - \alpha W(t)\right)$ Jointly Normal.

<u>Reason:</u> ① $W(t) \sim N(0, t)$.

② $W(t) - W(s) \sim N(0, t-s)$. & ind of $W(s)$.

③ $W(s) \sim N(0, s)$.

$$\Rightarrow \left(W(s),\ W(t) - W(s)\right) \sim N\left(0, \begin{pmatrix} s & 0 \\ 0 & t-s \end{pmatrix}\right).$$

$\left(W(t),\ W(s) - \alpha W(s)\right)$ = linear comb of

$\left(W(s),\ W(t) - W(s)\right) \Rightarrow$ Joint Normal